SUMS OF FOURTH POWERS AND RELATED TOPICS

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1. Introduction

The continuing renaissance in the theory and application of the Hardy-Littlewood method has produced significant advances in Waring's problem, in particular with respect to our understanding of sums of cubes, and sums of kth powers for larger k (see, for example, [16], [17], [18], [19], [22] and [24]). While important progress has been made concerning sums of kth powers for smaller k, recent improvements have been comparatively modest in scale, especially so far as fourth powers (biquadrates) are concerned. The object of this paper is to make further progress on such additive problems involving biquadrates. Recent developments elsewhere in Waring's problem have made use of strong new bounds for mean values of exponential sums over smooth numbers, and indeed these bounds are of utility in a plethora of additive problems. In contrast to the latter methods, the ideas underlying the conclusions of this paper make use only of an elementary polynomial identity, and are quite narrowly restricted in their application to additive problems involving several biquadrates. Nonetheless, despite the simplicity of our methods, we are able to tackle a number of problems which presently appear wholly beyond the reach of the more sophisticated machinery depending on the use of smooth numbers. The ideas presented below should therefore provide a useful addition to the arsenal of practitioners of the circle method.

The simplest consequence of our methods, which we deduce in §2, concerns the density of the set of integers represented as the sum of 5 biquadrates.

Theorem 1. *Let $N(X)$ denote the number of natural numbers up to X that can be written as the sum of 5 biquadrates. Then for each $\varepsilon > 0$ one has*

$$N(X) \gg X(\log X)^{-1-\varepsilon}.$$

The conclusion of Theorem 1 comes tantalisingly close to establishing that sums of 5 biquadrates have positive density. While it is conjectured that sums of 4 biquadrates have positive density, the best result along these lines available hitherto is that sums of 6 biquadrates have this property, such following directly from Vaughan [18, §§4 and 5]. Meanwhile, the lower bound $N(X) \gg X^{1-\delta}$, with $\delta = 0.0582\ldots$, follows from [18, Theorem 4.3], and would appear to be the best such bound easily available from the literature. We note, however, that slightly stronger bounds would follow, with sufficient effort, via the techniques of [20], [22] and [25].

As experts will instantly recognise, our high level of control over sums of 5 biquadrates, transparent from the conclusion of Theorem 1, leads to correspondingly powerful consequences for additive problems involving sums of 10 biquadrates, with cognate conclusions for problems involving 5 biquadrates. Following some preliminary work on the associated exponential sums in §3, we investigate the latter topics in §§4, 5 and 6. We begin, in §4, by considering sums of 10 biquadrates and a kth power.

Theorem 2. *Let k be a fixed natural number.*

(i) *When $4 \nmid k$, every sufficiently large integer can be written as the sum of 10 biquadrates and a kth power;*

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(ii) *When* $4 \mid k$, *every sufficiently large integer* n *satisfying* $n \equiv r \pmod{16}$ *with* $1 \le r \le 9$ *can be written as the sum of* 10 *biquadrates and a* k*th power.*

We note that when $k = 3$, the conclusion of part (i) of Theorem 2 is superseded by part (a)(i) of Theorem 5 below, and when $k \le 2$ this conclusion is weaker than results attainable easily through existing methods. For comparison, when $k \ge 5$ and $4 \nmid k$, the best methods available hitherto appear incapable even of demonstrating that all large integers are the sum of 11 biquadrates and a kth power. Of course, under the latter hypothesis on k, it is immediate from Vaughan [18, Theorem 1.2] that all large integers are the sum of 12 biquadrates and a kth power. When $4|k$ the problem of representing integers in the proposed manner is complicated by the local solubility condition arising from the prime 2, and a little thought reveals that when $n \equiv r \pmod{16}$, with $12 \le r \le 15$, then n cannot be written as the sum of 10 biquadrates and a kth power. Further, when $n \equiv 0 \pmod{16}$, it is relatively simple to find infinite families of integers, all divisible by 16, none of which can be written in the latter shape. The conclusion of Theorem 2(ii) therefore leaves open the question as to whether or not large integers in the residue classes 10 and 11 modulo 16 are represented in the proposed manner. While current philosophy would lead one to conjecture that such integers are indeed represented in this way, our methods contain an unfortunate artefact which in general entirely precludes their application to these latter congruence classes.

The situation in Theorem 2 of particular interest is that with $k = 4$, which is tantamount to Waring's problem for biquadrates. In §5, by appealing to methods of Vaughan [18], we are able to recover the congruence class 10 modulo 16 from the gap described in the previous paragraph. The remaining congruence class 11 modulo 16 is, unfortunately, entirely beyond the grasp of our method.

Theorem 3. *Every sufficiently large integer* n *satisfying* $n \equiv r \pmod{16}$ *with* $1 \le r \le 10$ *can be written as the sum of* 11 *biquadrates.*

We recall that Davenport [6] has shown that whenever $R \ge 14$, all large integers n with $n \equiv r \pmod{16}$ and $1 \le r \le R$ are the sum of R integral biquadrates, a conclusion successively improved on by Vaughan [17], [18], to the extent that the condition $R \ge 12$ is now known to be permissible. As is apparent, Theorem 3 narrowly misses showing that the latter condition can be weakened to $R \ge 11$.

As an easy consequence of the argument used to establish Theorem 2, we are able in §6 to establish related results concerning sums of 5 biquadrates and a kth power.

Theorem 4. *Let* k *be a fixed natural number.*

(i) *When* k *is odd, almost all natural numbers can be written as the sum of* 5 *biquadrates and a* k*th power;*

(ii) *When* $2|k$ *but* $4 \nmid k$, *almost all natural numbers* n *satisfying* $n \equiv r \pmod{16}$ *or* $n \equiv 8 + r \pmod{16}$, *with* $1 \le r \le 5$, *can be written as the sum of* 5 *biquadrates and a* k*th power;*

(iii) *When* $4|k$, *almost all natural numbers* n *satisfying* $n \equiv r \pmod{16}$ *with* $1 \le r \le 5$ *can be written as the sum of* 5 *biquadrates and a* k*th power.*

Once again, in the statement of Theorem 4, the congruence classes 6 and 14 modulo 16 in part (ii), and 6 modulo 16 in part (iii), are excluded from admissibility purely as an artefact of our method, and it is to be expected that the theorem should remain valid with their inclusion.

The scope for application of our methods to mixed additive problems involving biquadrates is great, and for the purposes of illustration we confine ourselves here to sums of cubes and biquadrates. In §§7 and 8 we establish the conclusions contained in the following theorem.

Theorem 5.

(a) *Every sufficiently large integer is represented in each of the following forms:*

(i) *as a sum of a cube and* 9 *biquadrates;*

(ii) *as a sum of* 2 *cubes and* 8 *biquadrates;*

(iii) *as a sum of* 3 *cubes and* 6 *biquadrates.*

(b) *Almost all natural numbers can be written as the sum of a cube and* 4 *biquadrates.*

Each of these conclusions is new, and apparently unattainable through existing methods. Meanwhile similar conclusions for 4 cubes and 5 biquadrates, 5 cubes and 3 biquadrates, and 6 cubes and 2 biquadrates, respectively, are attainable through existing technology (see, in particular, Brüdern [2]).

Before briefly describing the ideas underlying our method, we tarry a little longer to discuss applications of a more exotic flavour. First we remark on applications to the Waring-Goldbach problem for biquadrates. In §2 we make use of well-known lower bounds for the number of primes in 3-term arithmetic progressions to establish a lower bound similar to that provided in Theorem 1.

Theorem 6. *Let $N_1(X)$ denote the number of natural numbers n up to X that can be written in the form*

$$n = p_1^4 + p_2^4 + p_3^4 + p_4^4 + (2p_5)^4, \tag{1.1}$$

with the p_j prime numbers for $1 \le j \le 5$. Then for each $\varepsilon > 0$ one has

$$N_1(X) \gg X(\log X)^{-18-\varepsilon}.$$

Our analogue of Theorem 2 for prime numbers, which we establish in §9, involves a number of congruence conditions which, as is the case for Theorem 2 itself, are presumably not all necessary (we note, however, that if the primes used in the representation all exceed 5, then the congruence conditions are indeed necessary).

Theorem 7. *Let k be a fixed natural number with $k \ge 2$. Denote by $\mathcal{M}_k$ the set of integers defined by*

$$\mathcal{M}_k = \begin{cases} \{n \in \mathbb{N} : (n, 10) = 1 \text{ and } n \not\equiv 1 \pmod 3\}, & \text{when } k \text{ is odd,} \\ \{n \in \mathbb{N} : n \equiv 41 \text{ or } 89 \pmod{120}\}, & \text{when } 2|k \text{ but } 4 \nmid k, \\ \{n \in \mathbb{N} : n \equiv 41 \pmod{240}\}, & \text{when } 4|k. \end{cases}$$

Then every sufficiently large integer $n \in \mathcal{M}_k$ can be written in the form

$$n = \sum_{j=1}^{8} p_j^4 + (2p_9)^4 + (2p_{10})^4 + p_{11}^k, \tag{1.2}$$

with p_j prime for $1 \le j \le 11$.

Theorem 7 provides, in particular, an analogue of the Waring-Goldbach problem for biquadrates with 11 almost-prime summands. We note that when $k = 2$ we can reduce the number of biquadrates used in the representation (1.2) to 8.

The proofs of Theorems 1 to 7 are all based on the elementary polynomial identity

$$x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2, \tag{1.3}$$

which Dickson [9] attributes to F. Proth (see footnote 227 in Chapter XXII). The identity (1.3) permits us to specialise 3 biquadrates in such a way that their sum may be treated as a square. While it is true that the latter is in fact the square of the binary quadratic form $x^2 + xy + y^2$, the values assumed by this quadratic form are rather dense amongst the natural numbers, and thus we are able to bring into play the powerful apparatus from the Hardy-Littlewood method designed for handling mixed problems involving squares, biquadrates and so on (see the end of §2 for more general comments on the application of (1.3) in such a setting). As is evident, regrettably, the specialisation implicit in (1.3) constrains the sum of three biquadrates therein to be divisible by 2. Since the sum of three unconstrained biquadrates can occupy the residue classes 0, 1, 2 or 3 modulo 16, it is apparent that the use of (1.3) will necessarily impose additional congruence constraints within our applications, and indeed it is this observation which accounts for the "loss" of admissible residue classes in Theorems 2, 3 and 4. A second unfortunate consequence of our use of the specialisation implicit in (1.3) concerns the treatment of the major arcs in our applications of the Hardy-Littlewood method. Since, in essence, we are replacing 3 variables by one, the total number of variables available to us is significantly reduced. In particular, when it comes to bounding the exceptional sets arising in Theorems 4 and 5, we are forced to tackle the convergence of the singular series, and related auxiliary sums, for quaternary and ternary problems, respectively, the technical complexity of which is all too familiar to experts in this area. Such difficulties in large part account for the length of this memoir.

By a fortunate coincidence, the structure of the identity (1.3) provides for applications of more general type. Let $f(t)$ be a quartic polynomial of the shape $f(t) = at^4 + bt^2 + c$, and let $g(s)$ denote the quadratic polynomial $g(s) = 2as^2 + 2bs + 3c$. Then one has the identity

$$f(x) + f(y) + f(x + y) = g(x^2 + xy + y^2),$$

and thus with little difficulty one is able to adapt the methods of §§2 to 9 to handle additive problems involving polynomials of the shape $f(t)$. Of course, the difficulties associated with the congruence conditions inherent in such problems will differ from those involving pure biquadrates, sometimes to our advantage. For the purposes of illustration, we record without proof the following consequence of this circle of ideas.

Theorem 8. *Every sufficiently large integer n can be written in the form*

$$n = (x_1^4 - 2x_1^2) + (x_2^4 - 2x_2^2) + \cdots + (x_{11}^4 - 2x_{11}^2),$$

with $x_i \in \mathbb{N}$ $(1 \le i \le 11)$.

Since for a fixed h the polynomial $(h+x)^5 + (h-x)^5$ takes the special quartic shape discussed in the previous paragraph, the astute reader will anticipate the possibility of applying our ideas even to sums of fifth powers. We defer discussion of this prospect to a future memoir (see [13]), the applications being of a somewhat technical nature.

We finish by remarking that the methods of this paper are relevant to the study of $g(k)$ when $k = 4, 5$, where here, as usual, the function $g(k)$ denotes the least integer s such that all positive integers are the sum of s kth powers of non-negative integers. Formidable arguments of J.-R. Chen [5] and Balasubramanian, Deshouillers and Dress (see, in particular, [1] and [8]) have shown, respectively, that $g(5) = 37$ and $g(4) = 19$. The methods described herein allow alternative proofs to be provided for the latter conclusions, and indeed it is now possible to provide a significantly simpler proof that $g(4) = 19$. Moreover, when $s < g(k)$, one is also able to study the set of exceptional integers, with no representation as the sum of s kth powers of non-negative integers. This is a topic to which we intend to return elsewhere.

Throughout, the letter k denotes a fixed positive integer. We adopt the convention that whenever the letter ε appears in a statement, either explicitly or implicitly, then we assert that the statement holds for every sufficiently small positive number ε. The "value" of ε may consequently change from statement to statement. The implicit constants in Vinogradov's notation $\ll$ and $\gg$, and in Landau's notation, will depend at most on k and ε, unless stated otherwise. When x is a real number, we write $[x]$ for the greatest integer not exceeding x, and when n is an integer and p is a prime number we write $p^r \| n$ when $p^r | n$ but $p^{r+1} \nmid n$. We write $d(n)$ for the number of divisors of the integer n, and write also $\omega(n)$ for the number of distinct prime divisors of n. Finally, we adopt the convention throughout that any variable denoted by the letter p is implicitly assumed to be a prime number.

2. Sums of five biquadrates

We begin by exploring the consequences of the identity (1.3) for sums of five biquadrates, exploiting for this purpose well-known mean value estimates concerning two squares and four biquadrates (see, for example, Exercise 6 of [21, §2.8]).

The proof of Theorem 1. Let X be a large real number. Denote by $\mathcal{B}$ the set of integers of the form $x^2 + xy + y^2$, with $x, y \in \mathbb{Z}$. Also, when n is a natural number, let $r(n)$ denote the number of representations of n in the form

$$n = 2m^2 + u^4 + v^4, \tag{2.1}$$

with $m \in \mathcal{B}$ and $u, v \in \mathbb{N}$. Then in view of (1.3), whenever $r(n) > 0$, one has that n is the sum of 5 biquadrates, and thus, on recalling the notation of the statement of Theorem 1,

$$N(X) \ge \sum_{\substack{1 \le n \le X \\ r(n) > 0}} 1. \tag{2.2}$$

We next observe that when Y is large one has

$$\operatorname{card}(\mathcal{B} \cap [1, Y]) \gg Y (\log Y)^{-1/2}$$

(it is standard and well-known that an asymptotic formula holds; for rather general results of this type see, for example, [14]). Thus

$$\sum_{1 \le n \le X} r(n) \gg \sum_{1 \le u, v \le \frac{1}{2} X^{1/4}} \operatorname{card}\left(\mathcal{B} \cap [1, \tfrac{1}{2} X^{1/2}]\right) \gg X (\log X)^{-1/2}. \tag{2.3}$$

Moreover, the argument of the proof of Théorème 2′(i) of [15] (see §2, and in particular the estimation of W on p.235) provides the remarkably powerful estimate

$$\sum_{1 \le n \le X} r(n)^2 \ll X \exp\left((2+\varepsilon)\sqrt{(\log\log X)(\log\log\log X)}\right). \tag{2.4}$$

We may therefore apply Cauchy's inequality in standard fashion to conclude from (2.3) and (2.4) that

$$\sum_{\substack{1\le n\le X\\ r(n)>0}} 1 \ge \Big(\sum_{1\le n\le X} r(n)\Big)^2 \Big(\sum_{1\le n\le X} r(n)^2\Big)^{-1} \gg X(\log X)^{-1-\varepsilon},$$

and thus the theorem follows immediately from (2.2).

In order to establish Theorem 6 we augment the argument of the proof of Theorem 1 with a lower bound for the number of 3-term arithmetic progressions with prime entries lying in a fixed interval. Before embarking on the proof we first record some notation. When m is a natural number, denote by $\rho(m)$ the number of representations of m in the form $m = x^2 + xy + y^2$, with x, y and $\frac{1}{2}(x+y)$ all prime numbers. Define the set of integers $\mathcal{C}$ by

$$\mathcal{C} = \{m \in \mathbb{N} : \rho(m) > 0\}. \tag{2.5}$$

The proof of Theorem 6. Let X be a large real number. When n is a natural number, let $r(n)$ on this occasion denote the number of representations of n in the form (2.1) with $m \in \mathcal{C}$ and with u, v prime numbers. Then, again in view of (1.3), whenever $r(n) > 0$ one has that n is represented in the form (1.1), and thus, on recalling the notation of the statement of Theorem 6,

$$N_1(X) \ge \sum_{\substack{1\le n\le X\\ r(n)>0}} 1. \tag{2.6}$$

We first provide a lower bound for the cardinality of the set $\mathcal{C}\cap[1,X]$ for later use. The theory of the binary Goldbach problem (see [10], or [21, Chapter 3]) demonstrates that for each fixed $A > 0$, there is a fixed $B > 0$ such that for all large numbers x, all but at most $x(\log x)^{-A}$ of the integers h with $\frac{1}{2}x \le h \le x$ have at least $Bx(\log x)^{-2}$ representations in the form $2h = p_1 + p_2$, with p_i $(i = 1, 2)$ prime numbers. Consequently, for each large number x one has

$$\sum_{1\le m\le x} \rho(m) \ge \sum_{\substack{h \text{ prime}\\ 1\le h\le \frac{1}{2}x^{1/2}}} \sum_{\substack{p_1,p_2 \text{ primes}\\ p_1+p_2=2h}} 1 \gg x(\log x)^{-3}. \tag{2.7}$$

On the other hand, on writing $R(P)$ for the number of solutions of the diophantine equation

$$x_1^2 + x_1y_1 + y_1^2 = x_2^2 + x_2y_2 + y_2^2,$$

with $1 \le x_i, y_i \le P$ $(i = 1, 2)$, one has

$$\sum_{1\le m\le x} \rho(m)^2 \le R(x^{1/2}). \tag{2.8}$$

As an easy exercise one may establish the upper bound $R(P) \ll P^2\log(2P)$, and hence on combining Cauchy's inequality with (2.7) and (2.8) one obtains

$$\sum_{\substack{1\le m\le x\\ \rho(m)>0}} 1 \ge \Big(\sum_{1\le m\le x} \rho(m)\Big)^2 \Big(\sum_{1\le m\le x} \rho(m)^2\Big)^{-1} \gg x(\log x)^{-7}.$$

Thus, on recalling the definition of $r(n)$, one has

$$\sum_{1\le n\le X} r(n) \ge \sum_{\substack{u,v \text{ primes}\\ 1\le u,v\le \frac{1}{2}X^{1/4}}} \operatorname{card}\Big(\mathcal{C}\cap[1,\tfrac{1}{2}X^{1/2}]\Big) \gg X(\log X)^{-9}.$$

Consequently, since the upper bound (2.4) remains valid with the more restrictive definition of $r(n)$ holding here, the conclusion of Theorem 6 follows from (2.6) through an application of Cauchy's inequality paralleling that concluding the proof of Theorem 1.

The key ideas in the proof of Theorems 1 and 6 are susceptible to generalisation, as we now illustrate. When t is a natural number, let $w(t)$ denote a non-negative weight satisfying the condition that for each large number x one has

$$0<\sum_{1\le t\le x}w(t)\ll x^{\varepsilon}\sum_{1\le t\le x/2}w(t), \tag{2.9}$$

and for each $\varepsilon>0$,

$$\Bigl(\sum_{1\le t\le x}w(t)\Bigr)^2\gg x^{1/2-\varepsilon}\sum_{1\le t\le x}w(t)^2. \tag{2.10}$$

Let $N^*(x;w)$ denote the number of natural numbers up to x which can be written as the sum of 3 biquadrates and an integer t with $w(t)>0$. Then the argument used to establish Theorem 1 is easily adapted to establish the lower bound $N^*(x;w)\gg x^{1-\varepsilon}$. Seen from this perspective, a slightly weaker version of Theorem 1 is immediate on taking $w(t)$ to be the number of ways of writing t as the sum of two biquadrates. Other choices for $w(t)$ satisfying (2.9) and (2.10) may be lifted from the stock of examples familiar to additive number theorists. For example, one may take $w(t)$ to be the number of representations of t as the sum of a cube and a sixth power, or indeed the number of representations of t as the sum of a biquadrate, an eighth power, ..., a 2^{l-1}th power, and two 2^lth powers, for any fixed l with $l\ge 3$. There will be associated conclusions concerning the representation of integers as sums of 6 biquadrates, two integers t_1 and t_2 with $w(t_i)>0$ $(i=1,2)$ and a kth power, for example, although we stress that unwanted congruence conditions may be generated through the artificial nature of our construction.

3. Notation and Preliminaries

In advance of our various applications of the Hardy-Littlewood method in the remainder of this paper, we first record some notation, and also establish some auxiliary estimates associated with the exponential sums arising from an identity equivalent to (1.3), namely

$$(x+y)^4+(x-y)^4+(2y)^4=2(x^2+3y^2)^2. \tag{3.1}$$

Let N denote a sufficiently large positive integer. Further, when k is a positive integer, write

$$P_k=N^{\frac{1}{k}}. \tag{3.2}$$

We will frequently abbreviate P_4 simply to P. We write $e(z)$ for $e^{2\pi iz}$, and introduce the exponential sums

$$f_k(\alpha)=\sum_{1\le x\le P_k}e(\alpha x^k)\quad\text{and}\quad g(\alpha)=\sum_{\substack{P/4\le x,y\le P\\ x\ne y}}e(2(x^2+3y^2)^2\alpha). \tag{3.3}$$

We approximate the latter sums on the major arcs by means of the generating functions

$$S_k(q,a)=\sum_{r=1}^q e(ar^k/q),\quad S(q,a)=\sum_{r=1}^q\sum_{s=1}^q e(2a(r^2+3s^2)^2/q), \tag{3.4}$$

and

$$v_k(\beta)=\int_0^{P_k}e(\beta t^k)dt,\quad v(\beta)=\int_{P/4}^P\int_{P/4}^P e(2(t^2+3u^2)^2\beta)dtdu. \tag{3.5}$$

Lemma 3.1. *Suppose that $a\in\mathbb{Z}$, $q\in\mathbb{N}$ and $\alpha\in\mathbb{R}$ satisfy $(a,q)=1$ and $\alpha=\beta+a/q$. Then whenever $k\ge 2$ one has*

$$S_k(q,a)\ll q^{1-1/k},\quad v_k(\beta)\ll P_k(1+N|\beta|)^{-1/k},$$

and for each $\varepsilon>0$,

$$f_k(\alpha)-q^{-1}S_k(q,a)v_k(\beta)\ll q^{1/2+\varepsilon}(1+N|\beta|)^{1/2}.$$

Proof. For the proofs of these assertions, see, respectively, Theorems 4.2, 7.3 and 4.1 of [21].

It is convenient to record an estimate for $S_k(q,a)$ sharper than that provided by Lemma 3.1 for use in computations concerning the singular series.

Lemma 3.2. *Suppose that p is a prime number with $p > k$, and that a is an integer with $(p, a) = 1$. Then one has*

$$S_k(p, a) \le (k-1)p^{1/2}, \quad S_k(p^h, a) = p^{h-1} \quad (2 \le h \le k),$$

and

$$S_k(p^h, a) = p^{k-1} S_k(p^{h-k}, a) \quad (h > k).$$

Proof. These estimates are immediate from Lemmata 4.3 and 4.4 of [21].

We next investigate approximations to the exponential sum $g(\alpha)$.

Lemma 3.3. *Suppose that $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy $(a, q) = 1$ and $\alpha = \beta + a/q$. Then*

$$S(q, a) \ll q^{3/2} d(q), \quad v(\beta) \ll P^2(1 + N|\beta|)^{-1}, \tag{3.6}$$

where $d(q)$ denotes the number of divisors of q, and

$$g(\alpha) - q^{-2} S(q, a) v(\beta) \ll qP(1 + N|\beta|). \tag{3.7}$$

Proof. We begin by bounding $S(q, a)$ when $(a, q) = 1$, noting simply that by (3.4),

$$\begin{aligned} S(q, a) &= q^{-1} \sum_{n=1}^{q} e(2an^2/q) \sum_{l=1}^{q} \sum_{r=1}^{q} \sum_{s=1}^{q} e(l(r^2 + 3s^2 - n)/q) \\ &= q^{-1} \sum_{l=1}^{q} T(q, 2a, -l) T(q, l, 0) T(q, 3l, 0), \end{aligned} \tag{3.8}$$

where we write

$$T(q, c, d) = \sum_{y=1}^{q} e((cy^2 + dy)/q).$$

But for each pair of integers, c and d, one has

$$\begin{aligned} |T(q, c, d)|^2 &= \sum_{x=1}^{q} \sum_{y=1}^{q} e\left((c((x+y)^2 - x^2) + d((x+y) - x))/q \right) \\ &= \sum_{y=1}^{q} e((cy^2 + dy)/q) \sum_{x=1}^{q} e(2cxy/q) \\ &= q \sum_{\substack{y=1 \\ q|2cy}}^{q} e((cy^2 + dy)/q) \le q(q, 2c). \end{aligned}$$

Consequently, on recalling that $(a, q) = 1$, it follows from (3.8) that

$$S(q, a) \ll q^{1/2} \sum_{l=1}^{q} (q, l) \le q^{3/2} d(q),$$

and this establishes the first estimate of (3.6).

In order to estimate $v(\beta)$ we make use of the auxiliary estimate

$$\int_B^A \xi e(\gamma \xi^4) d\xi \ll \min\left\{ A^2, A^{-2} |\gamma|^{-1} \right\}, \tag{3.9}$$

valid for $A > B \gg A > 0$ and $\gamma \in \mathbb{R}$. In order to establish this bound, we note first that the left hand side of (3.9) is $O(A^2)$, by a trivial estimate. Moreover the change of variable $v = \xi^4$, followed by a partial integration, shows that the integral on the left hand side of (3.9) is $O(A^{-2}|\gamma|^{-1})$. Having established (3.9), we next observe that by a change of variables in (3.5), it suffices to establish the estimate

$$\int_{1/4}^{1}\int_{1/4}^{1} e(2(\xi^2+3\eta^2)^2\beta)d\xi d\eta \ll \min\{1,|\beta|^{-1}\}. \tag{3.10}$$

We dissect the square $\left[\frac{1}{4},1\right]^2$ into the triangular regions

$$\mathcal{B}_1=\{(\xi,\eta)\in\left[\tfrac{1}{4},1\right]^2:\xi\le\eta\} \quad\text{and}\quad \mathcal{B}_2=\{(\xi,\eta)\in\left[\tfrac{1}{4},1\right]^2:\xi>\eta\}.$$

By the change of variable $\xi=\eta\omega$, we obtain

$$\Big|\iint_{\mathcal{B}_1} e(2(\xi^2+3\eta^2)^2\beta)d\xi d\eta\Big|=\Big|\int_{1/4}^{1}\int_{(4\eta)^{-1}}^{1} \eta e(2(\omega^2+3)^2\eta^4\beta)d\omega d\eta\Big|,$$

whence, by interchanging the order of integration and making use of (3.9), we find that

$$\begin{aligned}\Big|\iint_{\mathcal{B}_1} e(2(\xi^2+3\eta^2)^2\beta)d\xi d\eta\Big| &= \Big|\int_{1/4}^{1}\int_{(4\omega)^{-1}}^{1} \eta e(2(\omega^2+3)^2\eta^4\beta)d\eta d\omega\Big| \\ &\ll \int_{0}^{1}\min\left\{1,(\omega^2+3)^{-2}|\beta|^{-1}\right\}d\omega. \end{aligned}\tag{3.11}$$

Moreover an easy estimate for the final integral in (3.11) reveals that it is

$$O\Big(\min\{1,|\beta|^{-1}\}\Big).$$

The contribution from the region $\mathcal{B}_2$ may be bounded similarly, on interchanging the roles of ξ and η, and thus (3.10) follows on combining the latter estimates.

Finally, we establish the approximation (3.7). We may suppose that $q\le P$, for otherwise (3.7) is trivial. Write $\phi(r,s)=2(r^2+3s^2)^2$, and observe that

$$g(\alpha)=\sum_{r=1}^{q}\sum_{s=1}^{q} e(a\phi(r,s)/q)U(r,s)+O(P), \tag{3.12}$$

where

$$U(r,s)=\sum_{\substack{P/4\le x\le P\\ x\equiv r \pmod q}}\ \sum_{\substack{P/4\le y\le P\\ y\equiv s \pmod q}} e(\phi(x,y)\beta).$$

But a standard application of the mean value theorem yields

$$U(r,s)=\int_{-u_0}^{u_1}\int_{-t_0}^{t_1} e\left(\phi(qt+r,qu+s)\beta\right)dtdu+O(q^{-1}P+q^{-1}P^5|\beta|),$$

where

$$t_0=\left[\left(\tfrac{1}{4}P-r\right)/q\right]-\tfrac{1}{2},\quad t_1=[(P-r)/q]+\tfrac{1}{2},$$

$$u_0=\left[\left(\tfrac{1}{4}P-s\right)/q\right]-\tfrac{1}{2},\quad u_1=[(P-s)/q]+\tfrac{1}{2}.$$

Consequently, a change of variables followed by an adjustment in the range of integration yields the estimate

$$U(r,s)-q^{-2}v(\beta)\ll q^{-1}P+q^{-1}P^5|\beta|. \tag{3.13}$$

The desired conclusion (3.7) therefore follows on substituting (3.13) into (3.12), and recalling (3.4).

It will be convenient in what follows to refer to a mean value estimate contained, in all essentials, in the upper bound (2.4).

Lemma 3.4. *For each $\varepsilon > 0$ one has*

$$\int_0^1 \left|g(\alpha)^2 f_4(\alpha)^4\right| d\alpha \ll N^{1+\varepsilon}. \tag{3.14}$$

Proof. On recalling (3.3) and considering the underlying diophantine equation, one finds that the integral on the left hand side of (3.14) is bounded above by the number of solutions of the equation

$$2(x_1^2+3y_1^2)^2+u_1^4+u_2^4=2(x_2^2+3y_2^2)^2+u_3^4+u_4^4, \tag{3.15}$$

with $1 \le x_i, y_i \le P$ $(i=1,2)$ and $1 \le u_j \le P$ $(1 \le j \le 4)$. By means of elementary divisor function estimates, the number of solutions of (3.15) with $u_1^4+u_2^4=u_3^4+u_4^4$ is $O(P^{4+\varepsilon})$. Meanwhile, when $u_1^4+u_2^4 \ne u_3^4+u_4^4$ one finds that both of the integers $(x_1^2+3y_1^2) \pm (x_2^2+3y_2^2)$ are divisors of $(u_1^4+u_2^4)-(u_3^4+u_4^4)$, whence for each fixed choice of the u_i there are $O(N^\varepsilon)$ possible choices for $x_i^2+3y_i^2$ $(i=1,2)$, and hence $O(N^{2\varepsilon})$ possible choices for x_i and y_i $(i=1,2)$. Since there are $O(P^4)$ available possibilities for the u_i $(1 \le i \le 4)$, we conclude that the total number of solutions of (3.15) is $O(P^{4+\varepsilon})$, and the lemma follows immediately.

We also record a second even simpler mean value estimate.

Lemma 3.5. *For each $\varepsilon > 0$ one has*

$$\int_0^1 |g(\alpha)|^4 d\alpha \ll N^{1+\varepsilon}. \tag{3.16}$$

Proof. On considering the underlying diophantine equations, it follows from (3.3) that the integral on the left hand side of (3.16) is bounded above by the number of solutions of the diophantine system

$$m_1^2-m_2^2=m_3^2-m_4^2, \tag{3.17}$$

$$m_i=x_i^2+3y_i^2 \quad (1 \le i \le 4), \tag{3.18}$$

with $1 \le x_i, y_i \le P$ $(1 \le i \le 4)$. Suppose first that x_i, y_i $(i=3,4)$ satisfy the condition that $m_3 \ne m_4$. Then by applying an elementary estimate for the divisor function, it follows from (3.17) that the number of possible choices for m_1 and m_2 is $O(P^\varepsilon)$, whence by (3.18) there are $O(P^\varepsilon)$ possible choices for x_j, y_j $(j=1,2)$. Thus the total number of solutions of this type is $O(P^{4+\varepsilon})$. When x_i, y_i $(i=3,4)$ satisfy the condition that $m_3=m_4$, moreover, one has

$$x_1^2+3y_1^2=x_2^2+3y_2^2 \quad \text{and} \quad x_3^2+3y_3^2=x_4^2+3y_4^2,$$

and again elementary divisor function estimates show that the number of solutions counted here is $O(P^{4+\varepsilon})$. Thus we conclude that the total number of solutions is $O(N^{1+\varepsilon})$, and the proof of the lemma is complete.

In advance of our imminent applications of the Hardy-Littlewood method in the forthcoming sections, it is convenient to record notation for a generic dissection. Let Q be a large real number. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, write

$$\mathfrak{M}_Q(q,a)=\{\alpha \in [0,1): |q\alpha-a| \le QN^{-1}\},$$

and take $\mathfrak{M}(Q)$ to be the union of the intervals $\mathfrak{M}_Q(q,a)$ with $0 \le a \le q \le Q$ and $(a,q)=1$. Note that when $Q < \frac{1}{2}N^{1/2}$, the intervals occurring in the latter union are disjoint. Finally, write $\mathfrak{m}(Q)=[0,1) \setminus \mathfrak{M}(Q)$.

4. Sums of 10 biquadrates and a kth power

Our preparations complete, we now apply the Hardy-Littlewood method to establish Theorem 2. Although many of the details will be considered by experts to be routine, we preserve some measure of completeness in our exposition for the edification of those less expert in the application of the circle method.

The proof of Theorem 2. Let k be a fixed natural number with $k \geq 2$, and let n be a large positive integer. Write $\mathcal{R}_k(n)$ for the number of representations of n in the form

$$n = \sum_{i=1}^{2} \left((x_i + y_i)^4 + (x_i - y_i)^4 + (2y_i)^4\right) + \sum_{j=1}^{4} z_j^4 + w^k, \tag{4.1}$$

with x_i, y_i natural numbers satisfying $x_i \neq y_i$ $(i = 1, 2)$, and with z_j $(1 \leq j \leq 4)$ and w natural numbers. We aim to establish that $\mathcal{R}_k(n) > 0$, whence, as is evident from (4.1), the integer n is represented as the sum of 10 biquadrates and a kth power.

It is convenient, for later use, to take N to be the large real parameter introduced in the previous section, and to consider an integer n with $N/2 < n \leq N$. When $\mathfrak{B} \subseteq [0, 1)$, define

$$\mathcal{R}_k(n; \mathfrak{B}) = \int_{\mathfrak{B}} g(\alpha)^2 f_4(\alpha)^4 f_k(\alpha) e(-n\alpha) d\alpha. \tag{4.2}$$

Write $X = P_{8k}$, $\mathfrak{M} = \mathfrak{M}(X)$ and $\mathfrak{m} = \mathfrak{m}(X)$. Then on recalling (3.1)-(3.3), it follows from orthogonality that

$$\mathcal{R}_k(n) \geq \mathcal{R}_k(n; [0, 1)) = \mathcal{R}_k(n; \mathfrak{M}) + \mathcal{R}_k(n; \mathfrak{m}). \tag{4.3}$$

The estimation of $\mathcal{R}_k(n; \mathfrak{m})$ is routine. By Weyl's inequality (see, for example, [21, Lemma 2.4]), one has

$$\sup_{\alpha \in \mathfrak{m}} |f_k(\alpha)| \ll P_k^{1+\varepsilon} X^{-2^{1-k}} \ll P_k N^{-2\delta}, \tag{4.4}$$

where $\delta = (k2^{k+4})^{-1}$. In view of (4.2), therefore, we deduce from Lemma 3.4 that

$$\begin{aligned} \mathcal{R}_k(n; \mathfrak{m}) &\leq \sup_{\alpha \in \mathfrak{m}} |f_k(\alpha)| \int_0^1 \left|g(\alpha)^2 f_4(\alpha)^4\right| d\alpha \\ &\ll P_k N^{1+\varepsilon-2\delta} \ll N^{1+1/k-\delta}. \end{aligned} \tag{4.5}$$

In order to estimate $\mathcal{R}_k(n; \mathfrak{M})$ we first introduce some additional notation. When l is a natural number, Q is a large real number with $Q < \frac{1}{2}N^{1/2}$, and $\alpha \in \mathbb{R}$, define $V_l(\alpha) = V_l(\alpha; Q)$ by

$$V_l(\alpha; Q) = \begin{cases} q^{-1} S_l(q, a) v_l(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}_Q(q, a) \subseteq \mathfrak{M}(Q), \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Similarly, define $W(\alpha) = W(\alpha; Q)$ by

$$W(\alpha; Q) = \begin{cases} q^{-2} S(q, a) v(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}_Q(q, a) \subseteq \mathfrak{M}(Q), \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

Then on combining trivial estimates for the relevant exponential sums together with the estimates provided by Lemmata 3.1 and 3.3, one has for each $\alpha \in \mathfrak{M}$ the upper bound

$$g(\alpha)^2 f_4(\alpha)^4 f_k(\alpha) - W(\alpha; X)^2 V_4(\alpha; X)^4 V_k(\alpha; X) \ll X^{1/2+\varepsilon} P^8 + X P^7 P_k.$$

But the measure of $\mathfrak{M}$ is plainly $O(X^2 N^{-1})$, so that by (4.2),

$$\begin{aligned} \mathcal{R}_k(n; \mathfrak{M}) - &\int_0^1 W(\alpha; X)^2 V_4(\alpha; X)^4 V_k(\alpha; X) e(-n\alpha) d\alpha \\ &\ll N^{-1} \left(X^{5/2+\varepsilon} P^8 + X^3 P^7 P_k\right) \ll N^{1+1/k-\delta}. \end{aligned} \tag{4.8}$$

It follows from (4.3), (4.5) and (4.8) that

$$\mathcal{R}_k(n; [0, 1)) - \sum_{1 \leq q \leq X} \mathcal{S}_k(q, n) J_k^*(q, n; N, X) \ll N^{1+1/k-\delta}, \tag{4.9}$$

where

$$\mathcal{S}_k(q,n) = \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-9} S(q,a)^2 S_4(q,a)^4 S_k(q,a) e(-na/q), \tag{4.10}$$

and

$$J_k^*(q,n;N,X) = \int_{-q^{-1}XN^{-1}}^{q^{-1}XN^{-1}} v(\beta)^2 v_4(\beta)^4 v_k(\beta) e(-n\beta)\, d\beta. \tag{4.11}$$

Next write

$$J_k(n) = \int_{-\infty}^{\infty} v(\beta)^2 v_4(\beta)^4 v_k(\beta) e(-\beta n)\, d\beta, \tag{4.12}$$

the absolute convergence of which is assured by means of Lemmata 3.1 and 3.3. By employing the latter lemmata one finds from (4.11) and (4.12) that

$$J_k^*(q,n;N,X) - J_k(n) \ll N^{2+1/k} \int_{q^{-1}XN^{-1}}^{\infty} (1+N\beta)^{-3-1/k}\, d\beta,$$

so that whenever $1 \le q \le X$ and $0 < \theta \le 2 + 1/k$ one has

$$J_k^*(q,n;N,X) - J_k(n) \ll N^{1+1/k} (q/X)^{\theta}. \tag{4.13}$$

Moreover, again by Lemmata 3.1 and 3.3,

$$J_k(n) \ll N^{2+1/k} \int_0^{\infty} (1+N\beta)^{-3-1/k}\, d\beta \ll N^{1+1/k}. \tag{4.14}$$

On recalling (4.10), we find from Lemmata 3.1 and 3.3 that one has

$$\mathcal{S}_k(q,n) \ll q^{\varepsilon-1-1/k}. \tag{4.15}$$

It therefore follows from (4.13) via yet another application of Lemmata 3.1 and 3.3 that

$$\begin{aligned} J_k(n) \sum_{1\le q\le X} \mathcal{S}_k(q,n) - \sum_{1\le q\le X} \mathcal{S}_k(q,n) J_k^*(q,n;N,X) \\ \ll N^{1+1/k} \sum_{1\le q\le X} q^{\varepsilon-1-1/k} (q/X)^{\theta}, \end{aligned}$$

whence, on taking $\theta = 1/(2k)$, we deduce that

$$J_k(n) \sum_{1\le q\le X} \mathcal{S}_k(q,n) - \sum_{1\le q\le X} \mathcal{S}_k(q,n) J_k^*(q,n;N,X) \ll N^{1+1/k-\delta}. \tag{4.16}$$

Finally, we write

$$\mathfrak{S}_k(n) = \sum_{q=1}^{\infty} q^{-9} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)^2 S_4(q,a)^4 S_k(q,a) e(-na/q).$$

On recalling (4.15), we find that

$$\mathfrak{S}_k(n) - \sum_{1\le q\le X} \mathcal{S}_k(q,n) \ll \sum_{q>X} q^{-1-1/(2k)} \ll X^{-1/(2k)}.$$

It therefore follows from (4.14) and (4.16) that

$$\mathfrak{S}_k(n) J_k(n) - \sum_{1\le q\le X} \mathcal{S}_k(q,n) J_k^*(q,n;N,X) \ll N^{1+1/k-\delta} + N^{1+1/k} X^{-1/(2k)},$$

whence by (4.9),

$$\mathcal{R}_k(n;[0,1)) - J_k(n)\mathfrak{S}_k(n) \ll N^{1+1/k-\delta}. \tag{4.17}$$

The estimation of the singular integral $J_k(n)$ is standard. By a change of variables one deduces from (4.12) that

$$J_k(n) = N^{1+1/k} \int_{-\infty}^{\infty} \int_{\mathcal{B}} e\left(\beta\left(\Phi(\boldsymbol{\gamma}) - n/N\right)\right) d\boldsymbol{\gamma} d\beta,$$

where $\mathcal{B} = [0,1]^5 \times \left[\frac{1}{4}, 1\right]^4$, and

$$\Phi(\boldsymbol{\gamma}) = \gamma_1^4 + \cdots + \gamma_4^4 + \gamma_5^k + 2(\gamma_6^2 + 3\gamma_7^2)^2 + 2(\gamma_8^2 + 3\gamma_9^2)^2. \tag{4.18}$$

Thus a straightforward application of Fourier's integral formula confirms that

$$J_k(n) \gg N^{1+1/k}. \tag{4.19}$$

It remains to analyse the singular series $\mathfrak{S}_k(n)$. It follows from the standard theory of exponential sums (see, for example, the proofs of Lemmata 2.10 and 2.11 of [21]) that $\mathcal{S}_k(q,n)$ is a multiplicative function of q. Moreover, by (4.15), the series

$$\mathfrak{S}_k(n) = \sum_{q=1}^{\infty} \mathcal{S}_k(q,n)$$

is absolutely convergent, and for each prime number p one has

$$\sum_{h=0}^{\infty} \mathcal{S}_k(p^h, n) = 1 + O(p^{-1-1/(2k)}). \tag{4.20}$$

The elementary theory of series of multiplicative functions consequently shows that

$$\mathfrak{S}_k(n) = \prod_p T_k(p,n), \tag{4.21}$$

where the product is over prime numbers, and

$$T_k(p,n) = \sum_{h=0}^{\infty} \mathcal{S}_k(p^h, n). \tag{4.22}$$

In order to handle the contribution of the small primes p in the product (4.21), we adapt the standard treatment used in Waring's problem, as described, for example, in [21, §2.6]. We observe first that by the argument of the proof of [21, Lemma 2.12], one has for each $H \geq 1$,

$$\sum_{h=0}^{H} \mathcal{S}_k(p^h, n) = p^{-8H} M_{k,n}(p^H), \tag{4.23}$$

where $M_{k,n}(q)$ denotes the number of solutions of the congruence

$$\Phi(\mathbf{z}) \equiv n \pmod{q},$$

with $\Phi(\mathbf{z})$ defined by (4.18), and with $1 \leq z_i \leq q$ $(1 \leq i \leq 9)$. It follows, in particular, that $T_k(p,n)$ is real and non-negative. By (4.20) and (4.21), moreover, there exists a positive absolute constant C with the property that

$$\tfrac{1}{2} \leq \prod_{p \geq C} T_k(p,n) \leq 2. \tag{4.24}$$

We aim to show, subject only to the condition that when $4|k$ one has $n \equiv r \pmod{16}$ with $1 \leq r \leq 9$, that for each prime p with $p < C$, one has

$$M_{k,n}(p^H) \gg p^{8H}, \tag{4.25}$$

with the implicit constant absolute. From the latter lower bound, by means of (4.21)-(4.24), it follows that $1 \ll \mathfrak{S}_k(n) \ll 1$, whence by (4.3), (4.17) and (4.19) we may conclude that $\mathcal{R}_k(n) \gg N^{1+1/k}$. This will complete the proof of Theorem 2.

Before advancing to establish the above claim (4.25) we pause to recall some of the standard theory from [21, §2.6], in a form appropriate to the application at hand. When p is a prime number, define $\gamma = \gamma(p)$ by

$$\gamma(p) = \begin{cases} 4, & \text{when } p = 2, \\ 1, & \text{otherwise.} \end{cases} \tag{4.26}$$

Then whenever a is a 4th power residue modulo p^γ, one has that a is a 4th power residue modulo p^t for every t. Moreover the number of 4th power residues modulo p^γ is $(p-1)/(4,p-1)$ when $p \neq 2$, and is precisely 1 when $p = 2$.

Consider first a prime number p with $p > 2$. Since for each natural number k, the monomial z^k represents 0 and 1 modulo p, the Cauchy-Davenport Theorem (see, for example, [21, Lemma 2.14]) shows that the number of distinct residue classes modulo p represented by the polynomial

$$y_1^4 + y_2^4 + y_3^4 + y_4^4 + w^k,$$

subject to $(y_1, p) = 1$, is at least

$$\min\left\{p\ ,\ 4\frac{p-1}{(4,p-1)} + 1\right\} = p.$$

Thus, for every integer n, when $p > 2$ there is a solution of the congruence

$$\Phi(\mathbf{z}) \equiv n \pmod{p^\gamma} \tag{4.27}$$

with $(z_1, p) = 1$. When $p = 2$ we argue directly. Observe that the polynomial $2(x^2 + 3y^2)^2$ represents the congruence classes 0 and 2 modulo 16. Also, when $4 \nmid k$, the set of values taken by the monomial w^k includes, at least, the residue classes 0, 1 and 9 modulo 16, and when $4|k$ the corresponding set consists only of 0 and 1 modulo 16. Then a little thought reveals that $\Phi(\mathbf{z})$ represents every residue class modulo 16 when $4 \nmid k$, and represents the residue class r modulo 16, for $1 \le r \le 9$, when $4|k$. Moreover, one may take z_1 to be odd in the latter representations. Thus, when $p = 2$ there is a solution of the congruence (4.27) with z_1 odd provided only that when $4|k$ one has $n \equiv r \pmod{16}$ with $1 \le r \le 9$.

Given a solution, $\mathbf{z}$, of the type described in the previous paragraph, and any natural number H, we generate a solution $\mathbf{x}$ to the congruence

$$\Phi(\mathbf{x}) \equiv n \pmod{p^H}$$

by choosing any integers x_i with $x_i \equiv z_i \pmod{p^\gamma}$ for $2 \le i \le 9$, and then solving the ensuing congruence modulo p^H for x_1. This congruence assumes the shape $x_1^4 \equiv m \pmod{p^H}$ with m a 4th power residue modulo p^γ, so is soluble by the above discussion. Since the number of such possible choices for $\mathbf{x}$ is evidently at least $p^{8(H-\gamma)}$, we deduce that for each prime p one has

$$M_{k,n}(p^H) \ge p^{8(H-\gamma)} \gg p^{8H},$$

thus confirming (4.25). This completes the proof of Theorem 2.

5. Sums of 11 biquadrates

In order to establish Theorem 3 we must resort to the use of smooth numbers. Before describing the proof of the latter theorem, it is useful to record some notation. When X and Y are positive real numbers, denote by $\mathcal{A}(X,Y)$ the set of Y-smooth numbers up to X, that is

$$\mathcal{A}(X,Y) = \{n \in [1,X] \cap \mathbb{Z}\ :\ p|n \text{ and } p \text{ prime implies that } p \le Y\}.$$

Write

$$h(\alpha; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^4).$$

Then it follows from Theorem 4.3 of Vaughan [18], together with the remark at the end of [24, §3], that when η is a sufficiently small positive number and $R \le P^\eta$, then

$$\int_0^1 |f_4(\alpha)^2 h(\alpha; P, R)^8| d\alpha \ll P^{6+\Delta}, \tag{5.1}$$

with Δ a certain positive number satisfying $\Delta \le 0.233$.

The proof of Theorem 3. Take N to be the large real parameter introduced in §3, and let η be a fixed positive number, sufficiently small in the context of the estimate (5.1). Consider an integer n with $N/2 < n \le N$, and let $\mathcal{R}(n)$ denote the number of representations of n in the form

$$n = (x+y)^4 + (x-y)^4 + (2y)^4 + \sum_{i=1}^4 z_i^4 + \sum_{j=1}^4 w_j^4,$$

with

$$1 \le z_i \le P \quad (1 \le i \le 4), \quad w_j \in \mathcal{A}(P, P^\eta) \quad (1 \le j \le 4),$$
$$P/4 \le x, y \le P \quad \text{and} \quad x \ne y.$$

We aim to show that whenever n is a large positive integer satisfying $n \equiv r \pmod{16}$ with $1 \le r \le 10$, then $\mathcal{R}(n) > 0$, whence n is represented as the sum of 11 biquadrates. The latter conclusion is tantamount to Theorem 3.

Abbreviate $h(\alpha; P, P^\eta)$ to $h(\alpha)$. Then by orthogonality one has

$$\mathcal{R}(n) = \int_0^1 g(\alpha) f_4(\alpha)^4 h(\alpha)^4 e(-n\alpha) d\alpha. \tag{5.2}$$

In order to estimate the integral in (5.2) we apply the Hardy-Littlewood method, dissecting the unit interval into the sets

$$\mathfrak{B}_1 = \mathfrak{m}(P/8), \quad \mathfrak{B}_2 = \mathfrak{M}(P/8) \setminus \mathfrak{M}(Y) \quad \text{and} \quad \mathfrak{B}_3 = \mathfrak{M}(Y),$$

where we write $Y = (\log P)^{1/4}$. Thus

$$\mathcal{R}(n) = R_1 + R_2 + R_3, \tag{5.3}$$

where

$$R_j = \int_{\mathfrak{B}_j} g(\alpha) f_4(\alpha)^4 h(\alpha)^4 e(-n\alpha) d\alpha \quad (j = 1, 2, 3). \tag{5.4}$$

The contribution, R_1, of the minor arcs to $\mathcal{R}(n)$ may be easily disposed of by appealing to Weyl's inequality (see, for example, [21, Lemma 2.4]). Thus, on applying Schwarz's inequality in combination with Lemma 3.4 and the inequality (5.1), and recalling that $P = N^{1/4}$, we obtain

$$\begin{aligned} R_1 &\le \sup_{\alpha \in \mathfrak{B}_1} |f_4(\alpha)| \left(\int_0^1 |g(\alpha)^2 f_4(\alpha)^4| d\alpha \right)^{1/2} \left(\int_0^1 |f_4(\alpha)^2 h(\alpha)^8| d\alpha \right)^{1/2} \\ &\ll P^{7/8+\varepsilon} \left(N^{1+\varepsilon} \right)^{1/2} \left(P^{6+\Delta} \right)^{1/2} \ll N^{3/2-\delta_1}, \end{aligned} \tag{5.5}$$

where $4\delta_1 = \frac{1}{8} - \frac{1}{2}\Delta - 3\varepsilon > 0.008$.

We estimate R_2 by applying Hölder's inequality and making use of the trivial estimate $|g(\alpha)| = O(P^2)$, thereby obtaining

$$R_2 \ll P^2 I_1^{2/3} I_2^{1/3}, \tag{5.6}$$

where

$$I_1 = \int_{\mathfrak{B}_2} |f_4(\alpha)|^6 d\alpha \quad \text{and} \quad I_2 = \int_{\mathfrak{M}(P/8)} |h(\alpha)|^{12} d\alpha. \tag{5.7}$$

It follows from the argument of the proof of [18, Lemma 5.1] that

$$I_1 \ll P^2 Y^{\varepsilon-1/4}. \tag{5.8}$$

Meanwhile, by considering the underlying diophantine equation one has

$$I_2 \leq \int_0^1 |h(\alpha)|^{12} d\alpha \leq \int_0^1 |f_4(\alpha)^4 h(\alpha)^8| d\alpha,$$

so that on applying the Hardy-Littlewood method we obtain

$$I_2 \leq I_3 + I_4, \tag{5.9}$$

where

$$I_3 = \int_{\mathfrak{M}(P/8)} |f_4(\alpha)^4 h(\alpha)^8| d\alpha \quad \text{and} \quad I_4 = \int_{\mathfrak{m}(P/8)} |f_4(\alpha)^4 h(\alpha)^8| d\alpha.$$

But by Weyl's inequality (see [21, Lemma 2.4]) together with (5.1), we have

$$\begin{aligned} I_4 &\leq \Big(\sup_{\alpha \in \mathfrak{m}(P/8)} |f_4(\alpha)| \Big)^2 \int_0^1 |f_4(\alpha)^2 h(\alpha)^8| d\alpha \\ &\ll \left(P^{7/8+\varepsilon} \right)^2 P^{6+\Delta} \ll P^{8-\delta_2}, \end{aligned} \tag{5.10}$$

where $\delta_2 = \frac{1}{4} - \Delta - 2\varepsilon > 0.016$. Moreover, on recalling (5.7), it follows from Hölder's inequality that

$$I_3 \leq \Big(\int_{\mathfrak{M}(P/8)} |f_4(\alpha)|^{12} d\alpha \Big)^{1/3} I_2^{2/3}.$$

Then by (5.9) and (5.10), together with an application of [18, Lemma 5.1], we deduce that

$$I_2 \leq P^{8-\delta_2} + (P^8)^{1/3} I_2^{2/3},$$

whence $I_2 \ll P^8$. On recalling (5.6) and (5.8), therefore, we conclude that

$$R_2 \ll P^2 (P^2 Y^{\varepsilon-1/4})^{2/3} (P^8)^{1/3} \ll N^{3/2} (\log N)^{-1/25}. \tag{5.11}$$

It remains only to estimate R_3. We begin with a little notation. Let $\rho(x)$ denote Dickman's function, defined for real x by

$$\rho(x) = 0 \quad \text{when} \quad x \leq 0,$$

$$\rho(x) = 1 \quad \text{when} \quad 0 < x \leq 1,$$

$$\rho \text{ is continuous for } x > 0,$$

$$\rho \text{ is differentiable for } x > 1,$$

$$x\rho'(x) = -\rho(x-1) \quad \text{for} \quad x > 1.$$

Define

$$w(\beta) = \int_{P^\eta}^P \rho\left(\frac{\log \gamma}{\eta \log P} \right) e(\beta\gamma^4) d\gamma,$$

and note that on following the argument sketched in the proof of [23, Lemma 8.6], one has

$$w(\beta) \ll P \left(1 + P^4|\beta|\right)^{-1/4}. \tag{5.12}$$

Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$, and write $\beta = \alpha - a/q$. Then [23, Lemma 8.5] shows that when $q \leq P^\eta$, one has

$$h(\alpha) - q^{-1} S_4(q,a) w(\beta) \ll \frac{qP}{\log P} \left(1 + P^4|\beta|\right). \tag{5.13}$$

When Q is a large real number, and $\alpha \in \mathbb{R}$, define $U(\alpha) = U(\alpha; Q)$ by

$$U(\alpha; Q) = \begin{cases} q^{-1}S_4(q,a)w(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}_Q(q,a) \subseteq \mathfrak{M}(Q), \\ 0, & \text{otherwise,} \end{cases}$$

and define $V_l(\alpha; Q)$ and $W(\alpha; Q)$ as in (4.6) and (4.7). Then on combining trivial estimates for the relevant exponential sums together with the estimates provided by Lemmata 3.1, 3.3 and (5.13), one has for each $\alpha \in \mathfrak{B}_3$ the upper bound

$$\begin{aligned} g(\alpha)f_4(\alpha)^4h(\alpha)^4 - W(\alpha;Y)V_4(\alpha;Y)^4U(\alpha;Y)^4 &\ll \frac{YP}{\log P}P^9 + YP^9 \\ &\ll YN^{5/2}(\log N)^{-1}. \end{aligned}$$

But the measure of $\mathfrak{B}_3$ is plainly $O(Y^2N^{-1})$, so that by (5.4) one has

$$\begin{aligned} R_3 - \int_0^1 &W(\alpha;Y)V_4(\alpha;Y)^4U(\alpha;Y)^4e(-n\alpha)d\alpha \\ &\ll Y^3N^{3/2}(\log N)^{-1} \ll N^{3/2}(\log N)^{-1/4}. \end{aligned} \tag{5.14}$$

Then on making use of (5.3), (5.5), (5.11) and (5.14), together with the definitions of W, V_4 and U, we may conclude thus far that

$$\mathcal{R}(n) - \sum_{1 \le q \le Y} \mathcal{S}(q,n)J^*(q,n;N,Y) \ll N^{3/2}(\log N)^{-1/25}, \tag{5.15}$$

where

$$\mathcal{S}(q,n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-10}S(q,a)S_4(q,a)^8e(-na/q),$$

and

$$J^*(q,n;N,Y) = \int_{-q^{-1}YN^{-1}}^{q^{-1}YN^{-1}} v(\beta)v_4(\beta)^4w(\beta)^4e(-n\beta)d\beta.$$

Next we write

$$J(n) = \int_{-\infty}^{\infty} v(\beta)v_4(\beta)^4w(\beta)^4e(-n\beta)d\beta, \tag{5.16}$$

and

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} q^{-10} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(q,a)S_4(q,a)^8e(-na/q). \tag{5.17}$$

By applying Lemmata 3.1, 3.3 and (5.12) to (5.16) and (5.17), it follows easily that $J(n) \ll N^{3/2}$ and $\mathfrak{S}(n) \ll 1$. Further, by following the argument applied in the proof of Theorem 2 leading to (4.17), mutatis mutandis, one arrives at the conclusion

$$J(n)\mathfrak{S}(n) - \sum_{1 \le q \le Y} \mathcal{S}(q,n)J^*(q,n;N,Y) \ll N^{3/2}Y^{-1/4},$$

whence by (5.15),

$$\mathcal{R}(n) - J(n)\mathfrak{S}(n) \ll N^{3/2}(\log N)^{-1/25}. \tag{5.18}$$

The estimation of the singular integral is complicated by the presence of weights, specifically, the implicit occurrence of the Dickman function. However, one may apply the argument of the proof of [26, Lemma 9.10] to obtain the lower bound

$$J(n) \gg N^{3/2}. \tag{5.19}$$

The singular series may be handled by the corresponding argument in the proof of Theorem 2. A little thought reveals that in order to establish the lower bound

$$\mathfrak{S}(n) \gg 1, \tag{5.20}$$

all that remains is to check that the congruence

$$2(x^2+3y^2)^2 + \sum_{j=1}^{8} z_j^4 \equiv n \pmod{16}$$

possesses a solution with z_1 odd for those n satisfying $n \equiv r \pmod{16}$ for some r with $1 \le r \le 10$. But since $2(x^2+3y^2)^2$ takes the values 0 and 2 modulo 16, the latter is easily checked by hand. Collecting together (5.18)-(5.20), we conclude that whenever n is a large integer with $N/2 < n \le N$, satisfying $n \equiv r \pmod{16}$ for some r with $1 \le r \le 10$, then one has $\mathcal{R}(n) \gg N^{3/2}$, and this completes the proof of Theorem 3.

6. Sums of 5 biquadrates and a kth power

The proof of Theorem 4 follows from the methods of the proof of Theorem 2 via an application of Bessel's inequality. A somewhat serious difficulty concerns the convergence of the singular series, and this issue forces us to modify the endgame of our argument in several important respects.

The proof of Theorem 4. Let k be a fixed natural number with $k \ge 2$, and let n be a large positive integer. Write $\mathcal{R}_k^*(n)$ for the number of representations of n in the form

$$n = (x+y)^4 + (x-y)^4 + (2y)^4 + z_1^4 + z_2^4 + w^k, \tag{6.1}$$

with x, y natural numbers satisfying $x \ne y$, and with z_j $(j = 1, 2)$ and w natural numbers. We aim to establish that when X is a large real number, then $\mathcal{R}_k^*(n) > 0$ for every n lying in a certain collection of congruence classes, and satisfying $1 \le n \le X$, with at most $o(X)$ possible exceptions. From this assertion, as is evident from (6.1), almost all integers n in the aforementioned congruence classes are represented as the sum of 5 biquadrates and a kth power.

We take N to be the large real parameter introduced in §3, and consider an integer n with $N/2 < n \le N$. When $\mathfrak{B} \subseteq [0, 1)$, define

$$\mathcal{R}_k^*(n; \mathfrak{B}) = \int_{\mathfrak{B}} g(\alpha) f_4(\alpha)^2 f_k(\alpha) e(-n\alpha)\, d\alpha. \tag{6.2}$$

Write $X = P_{8k}$, $\mathfrak{M} = \mathfrak{M}(X)$ and $\mathfrak{m} = \mathfrak{m}(X)$. Then on recalling (3.3), it follows from orthogonality that

$$\mathcal{R}_k^*(n) \ge \mathcal{R}_k^*(n; [0,1)) = \mathcal{R}_k^*(n; \mathfrak{M}) + \mathcal{R}_k^*(n; \mathfrak{m}). \tag{6.3}$$

We estimate $\mathcal{R}_k^*(n; \mathfrak{m})$ in mean square by applying Bessel's inequality, thereby obtaining

$$\sum_{N/2<n\le N} |\mathcal{R}_k^*(n; \mathfrak{m})|^2 \le \int_{\mathfrak{m}} |g(\alpha) f_4(\alpha)^2 f_k(\alpha)|^2 d\alpha.$$

Thus, on recalling (4.4), we deduce from Lemma 3.4 that

$$\begin{aligned}\sum_{N/2<n\le N} |\mathcal{R}_k^*(n; \mathfrak{m})|^2 &\le \Big(\sup_{\alpha\in\mathfrak{m}} |f_k(\alpha)|\Big)^2 \int_0^1 |g(\alpha)^2 f_4(\alpha)^4| d\alpha \\ &\ll (P_k N^{-2\delta})^2 N^{1+\varepsilon} \ll N^{1+2/k-\delta},\end{aligned} \tag{6.4}$$

where $\delta = (k2^{k+4})^{-1}$.

We next treat $\mathcal{R}_k^*(n; \mathfrak{M})$. Recall the definitions (4.6) and (4.7). Then on combining trivial estimates for the relevant exponential sums together with the estimates provided by Lemmata 3.1 and 3.3, one finds that for each $\alpha \in \mathfrak{M}$,

$$g(\alpha) f_4(\alpha)^2 f_k(\alpha) - W(\alpha; X) V_4(\alpha; X)^2 V_k(\alpha; X) \ll X^{1/2+\varepsilon} P^4 + X P^3 P_k.$$

But the measure of $\mathfrak{M}$ is plainly $O(X^2N^{-1})$, so that

$$\begin{aligned}\mathcal{R}_k^*(n;\mathfrak{M}) - \int_0^1 & W(\alpha;X)V_4(\alpha;X)^2V_k(\alpha;X)e(-n\alpha)d\alpha \\ & \ll N^{-1}\left(X^{5/2+\varepsilon}P^4 + X^3P^3P_k\right) \ll N^{1/k-\delta}.\end{aligned}$$

We therefore deduce that

$$\mathcal{R}_k^*(n;\mathfrak{M}) - \sum_{1\le q\le X} \mathcal{S}_k^*(q,n)J_k^*(q,n;N,X) \ll N^{1/k-\delta}, \tag{6.5}$$

where

$$\mathcal{S}_k^*(q,n) = \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-5}S(q,a)S_4(q,a)^2S_k(q,a)e(-na/q), \tag{6.6}$$

and

$$J_k^*(q,n;N,X) = \int_{-q^{-1}XN^{-1}}^{q^{-1}XN^{-1}} v(\beta)v_4(\beta)^2v_k(\beta)e(-n\beta)d\beta. \tag{6.7}$$

Next write

$$J_k^*(n) = \int_{-\infty}^{\infty} v(\beta)v_4(\beta)^2v_k(\beta)e(-n\beta)d\beta, \tag{6.8}$$

the absolute convergence of which is assured by means of Lemmata 3.1 and 3.3. By employing the latter estimates one finds from (6.7) that whenever $1 \le q \le X$ and $0 < \theta \le \frac{1}{2} + \frac{1}{k}$, one has

$$J_k^*(q,n;N,X) - J_k^*(n) \ll N^{1+1/k}\int_{q^{-1}XN^{-1}}^{\infty} (1+N\beta)^{-\frac{3}{2}-\frac{1}{k}}d\beta \ll N^{1/k}(q/X)^{\theta}.$$

Consequently, one has

$$\begin{aligned}J_k^*(n)\sum_{1\le q\le X}\mathcal{S}_k^*(q,n) - & \sum_{1\le q\le X}\mathcal{S}_k^*(q,n)J_k^*(q,n;N,X) \\ & \ll N^{1/k}\sum_{1\le q\le X}(q/X)^{\theta}|\mathcal{S}_k^*(q,n)|.\end{aligned} \tag{6.9}$$

Moreover, again by Lemmata 3.1 and 3.3,

$$J_k^*(n) \ll N^{1/k}. \tag{6.10}$$

In order to analyse the right hand side of (6.9), we note that the standard theory of exponential sums reveals that $\mathcal{S}_k^*(q,n)$ is multiplicative (see, for example, the proofs of Lemmata 2.10 and 2.11 of [21]). Thus

$$|\mathcal{S}_k^*(q,n)| = \prod_{p^h\| q} |\mathcal{S}_k^*(p^h,n)|. \tag{6.11}$$

Next we note that whenever $(q,t)=1$, then by a change of variables one has $S(q,a) = S(q,at^{4k})$, and similarly $S_4(q,a) = S_4(q,at^{4k})$ and $S_k(q,a) = S_k(q,at^{4k})$. On substituting the latter into (6.6), substituting a for occurrences of at^{4k}, we deduce that $\mathcal{S}_k^*(q,n) = \mathcal{S}_k^*(q,l^{4k}n)$, where l satisfies $lt \equiv 1 \pmod{q}$. Consequently, on summing over the values of l with $(l,q)=1$, we deduce that

$$\begin{aligned}\phi(q)\mathcal{S}_k^*(q,n) &= \sum_{\substack{l=1\\(l,q)=1}}^{q} \mathcal{S}_k^*(q,l^{4k}n) \\ &= q^{-5}\sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)S_4(q,a)^2S_k(q,a)U(q,-an),\end{aligned} \tag{6.12}$$

where we write

$$U(q,b)=\sum_{\substack{l=1\\(l,q)=1}}^{q} e(bl^{4k}/q).$$

Plainly,

$$U(q,b)=(q,b)U\left(\frac{q}{(q,b)},\frac{b}{(q,b)}\right). \tag{6.13}$$

Moreover, Lemma 1.2 of Hua [11] shows that whenever p is a prime number, h is a natural number, and b is an integer with $(b,p)=1$, then one has

$$U(p^h,b)\ll p^{h/2}.$$

Consequently, for each prime p and natural number h, it follows from (6.13) that whenever $(a,p)=1$, one has

$$U(p^h,-an)\ll p^{h/2}(p^h,n)^{1/2}. \tag{6.14}$$

On combining (6.14) with the estimates provided by Lemmata 3.2 and 3.3, it follows from (6.12) that, for each prime p and each natural number h, one has

$$\mathcal{S}_k^*(p^h,n)\ll hp^{-1-h/2}(p^h,n)^{1/2}, \tag{6.15}$$

whence by the multiplicative property (6.11) of $\mathcal{S}_k^*(q,n)$, we deduce that

$$\mathcal{S}_k^*(q,n)\ll \tilde{q}^{-1}q^{\varepsilon-1/2}(q,n)^{1/2}, \tag{6.16}$$

where $\tilde{q}$ denotes the squarefree kernel of q, that is

$$\tilde{q}=\prod_{p|q}p.$$

On substituting (6.16) into (6.9), we conclude that

$$\sum_{N/2<n\le N}\Bigl|J_k^*(n)\sum_{1\le q\le X}\mathcal{S}_k^*(q,n)-\sum_{1\le q\le X}\mathcal{S}_k^*(q,n)J_k^*(q,n;N,X)\Bigr|^2\ll N^{2/k}E(N,X), \tag{6.17}$$

where

$$E(N,X)=X^{-2\theta}\sum_{1\le n\le N}\sum_{1\le q_1,q_2\le X}(\tilde{q}_1\tilde{q}_2)^{-1}(q_1q_2)^{\theta+\varepsilon-1/2}(q_1,n)^{1/2}(q_2,n)^{1/2}. \tag{6.18}$$

But an elementary argument provides the estimate

$$\sum_{1\le n\le N}(q_1,n)^{1/2}(q_2,n)^{1/2}\le d(q_1)d(q_2)N, \tag{6.19}$$

and hence, on taking $\theta=1/6$, one finds from (6.18) together with an elementary estimate for the divisor function that

$$E(N,X)\ll NX^{-1/3}\Bigl(\sum_{1\le q\le X}\tilde{q}^{-1}q^{-1/6}\Bigr)^2. \tag{6.20}$$

Moreover,

$$\sum_{q=1}^{\infty}\tilde{q}^{-1}q^{-1/6}\le\prod_p\Bigl(1+\sum_{h=1}^{\infty}p^{-1-h/6}\Bigr)\ll 1. \tag{6.21}$$

We now collect together (6.3)-(6.5), (6.17), (6.20) and (6.21) to conclude thus far that

$$\sum_{N/2<n\le N}\Bigl|\mathcal{R}_k^*(n;[0,1))-J_k^*(n)\sum_{1\le q\le X}\mathcal{S}_k^*(q,n)\Bigr|^2\ll N^{1+2/k-\delta}+N^{1+2/k}X^{-1/3}. \tag{6.22}$$

Next we complete the singular series. Write

$$\mathfrak{S}_k^*(n)=\sum_{q=1}^{\infty}\mathcal{S}_k^*(q,n), \tag{6.23}$$

and

$$\mathcal{E}(n,X)=\mathfrak{S}_k^*(n)-\sum_{1\le q\le X}\mathcal{S}_k^*(q,n). \tag{6.24}$$

Then on applying the estimates (6.10), (6.16), (6.19) and (6.21) we obtain from (6.23) the upper bound

$$\begin{aligned}\sum_{N/2<n\le N}|\mathcal{E}(n,X)J_k^*(n)|^2&\ll N^{2/k}\sum_{N/2<n\le N}\Big|\sum_{q>X}(q/X)^{1/6}\mathcal{S}_k^*(q,n)\Big|^2\\&\ll N^{2/k}X^{-1/3}\sum_{1\le n\le N}\sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}(\tilde{q}_1\tilde{q}_2)^{-1}(q_1q_2)^{\varepsilon-1/3}(q_1,n)^{1/2}(q_2,n)^{1/2}\\&\ll N^{1+2/k}X^{-1/3}.\end{aligned}$$

We therefore conclude from (6.22) and (6.24) that

$$\sum_{N/2<n\le N}|\mathcal{R}_k^*(n;[0,1))-\mathfrak{S}_k^*(n)J_k^*(n)|^2\ll N^{1+2/k-\delta}. \tag{6.25}$$

The estimation of the singular integral defined in (6.8) may be completed as in the argument of the proof of Theorem 2 leading to (4.19). Thus we obtain for each n satisfying $N/2<n\le N$ the lower bound

$$J_k^*(n)\gg N^{1/k}. \tag{6.26}$$

We analyse the singular series in a manner also similar to the corresponding treatment in the argument of the proof of Theorem 2. First note that by (6.16) and (6.23) one has

$$\begin{aligned}\mathfrak{S}_k^*(n)-\sum_{1\le q\le X}\mathcal{S}_k^*(q,n)&\ll\sum_{q>X}(q/X)^{1/6}\mathcal{S}_k^*(q,n)\\&\ll X^{-1/6}\sum_{q>X}\tilde{q}^{-1}q^{\varepsilon-1/3}(q,n)^{1/2}\\&\ll n^{1/2}X^{-1/6},\end{aligned}$$

so that the series $\mathfrak{S}_k^*(n)$ is absolutely convergent. Further, from (6.15) one has that for each prime number p,

$$\sum_{h=0}^{\infty}\mathcal{S}_k^*(p^h,n)=1+O(n^{1/2}p^{-3/2}),$$

so that we may use the standard theory of series of multiplicative functions to conclude that

$$\mathfrak{S}_k^*(n)=\prod_p T_k^*(p,n), \tag{6.27}$$

where

$$T_k^*(p,n)=\sum_{h=0}^{\infty}\mathcal{S}_k^*(p^h,n). \tag{6.28}$$

Write

$$\Phi^*(\boldsymbol{\gamma})=\gamma_1^4+\gamma_2^4+\gamma_3^k+2(\gamma_4^2+3\gamma_5^2)^2,$$

and denote by $M_{k,n}^*(q)$ the number of solutions of the congruence

$$\Phi^*(\mathbf{z})\equiv n\pmod{q},$$

with $1 \le z_i \le q$ $(1 \le i \le 5)$. Then by the argument of the proof of [21, Lemma 2.12], one has for each $H \ge 1$,

$$\sum_{h=0}^{H} \mathcal{S}_k^*(p^h, n) = p^{-4H} M_{k,n}^*(p^H). \tag{6.29}$$

It follows that $T_k^*(p,n)$ is real and non-negative, whence the same holds for $\mathfrak{S}_k^*(n)$.

Next consider the contribution to $\mathfrak{S}_k^*(n)$ from those primes p in the product (6.27) with $p \nmid n$. Then from (6.28) and (6.15) we have

$$T_k^*(p,n) = 1 + O(p^{-3/2}),$$

whence, for some positive constant C, one has

$$\tfrac{1}{2} \le \prod_{\substack{p \ge C \\ p \nmid n}} T_k^*(p,n) \le 2. \tag{6.30}$$

When $p > 3$, we may apply the Cauchy-Davenport Theorem to obtain useful bounds. Note first that when $p \nmid m$, one has

$$\sum_{y=1}^{p} \left(1 + \left(\frac{m - 3y^2}{p} \right) \right) = p - \left(\frac{-3}{p} \right) \ge 4,$$

so that every non-zero residue class m is represented by the form $x^2 + 3y^2$, and moreover the zero residue class is plainly represented in the latter form. Next observe that for each natural number k, the monomial z^k represents 0 and 1 modulo p. Then the Cauchy-Davenport Theorem (see, for example, [21, Lemma 2.14]) shows that the number of distinct residue classes modulo p represented by the polynomial

$$y_1^4 + y_2^4 + y_3^k + 2w^2, \tag{6.31}$$

subject to $(y_1, p) = 1$, is at least

$$\min \left\{ p \ , \ 2\frac{p-1}{(4, p-1)} + \tfrac{1}{2}(p-1) + 1 \right\} = p.$$

Thus every residue class modulo p is represented by the polynomial (6.31) with $(y_1, p) = 1$, so that in view of our earlier observation, it follows that for every integer n, when $p > 3$ there is a solution of the congruence

$$\Phi^*(\mathbf{z}) \equiv n \pmod{p^\gamma},$$

with $(z_1, p) = 1$. Here γ is defined as in (4.26). Observe also that the polynomial $y_1^4 + y_2^4 + 2(u^2 + 3v^2)^2$ plainly represents all residue classes modulo 3 with $(y_1, 3) = 1$. Further, we note that the polynomial $2(x^2 + 3y^2)^2$ represents the congruence classes 0 and 2 modulo 16. Also, when $2 \nmid k$, the set of values taken by the monomial w^k includes, at least, the residue classes 0 and $2r - 1$ modulo 16 $(1 \le r \le 8)$, and when $2 \| k$ the set of values includes, at least, the residue classes 0, 1 and 9 modulo 16, and when $4 | k$ the corresponding set consists only of 0 and 1 modulo 16. Then a little thought reveals that $\Phi^*(\mathbf{z})$ represents every residue class modulo 16 when $2 \nmid k$, represents the residue classes r and $8 + r$ modulo 16 for $1 \le r \le 5$ when $2 \| k$, and when $4 | k$ represents the residue class r modulo 16 for $1 \le r \le 5$. Furthermore, in all of these representations except for the representation of the residue class 9 modulo 16 when $2 \| k$, one may take z_1 to be odd. Moreover, in the latter exceptional case one may take z_3 to be odd.

Then in all cases, with the aforementioned exception, one may follow the argument completing the proof of Theorem 2 to conclude that whenever n lies in the relevant congruence classes modulo 16, one has for every prime p and natural number H,

$$M_{k,n}^*(p^H) \ge p^{4(H-\gamma)} \gg p^{4H},$$

whence (6.28) and (6.29) together show that

$$T_k^*(p,n) = \sum_{h=0}^{\infty} \mathcal{S}_k^*(p^h, n) \gg 1. \tag{6.32}$$

Moreover, in the exceptional case one may again argue as in the completion of the proof of Theorem 2, save that we now choose integers x_i with $x_i \equiv z_i \pmod{2^\gamma}$ for $i = 1, 2, 4, 5$, and then solve the ensuing congruence $\Phi(\mathbf{x}) \equiv n \pmod{2^H}$ for x_3. Thus, with modest adjustments to the argument, one again establishes the lower bound (6.32) even in the exceptional case under consideration. Combining the lower bound (6.32) with (6.30), we conclude that there is an absolute constant $A > 0$ such that

$$\mathfrak{S}_k^*(n) \geq \tfrac{1}{2} A^C \prod_{p|n} A \gg A^{\omega(n)} \gg N^{-\delta/4}.$$

Finally, we recall (6.25) and (6.26), and conclude that for each n with $N/2 < n \leq N$ in the aforementioned residue classes modulo 16, one has

$$\mathcal{R}_k^*(n; [0, 1)) \gg N^{1/k - \delta/4}$$

with at most $O(N^{1-\delta/2})$ possible exceptions. On summing over the dyadic intervals spanning $[1, N]$, and noting (6.3), we find that Theorem 4 follows.

7. Mixed sums of cubes and biquadrates

In this section we establish the results contained in part (a) of Theorem 5, and we also prepare the field for our assault on the proof of part (b) in §8 below. We begin by dismissing part (iii) of Theorem 5(a) almost trivially by recourse to Brüdern [2, Theorem 1]. Consider a large natural number N, and let $r(N)$ denote the number of distinct integers of the form $N - \sum_{i=1}^5 m_i^4$ with $1 \leq m_j \leq \frac{1}{2} N^{1/4}$ $(1 \leq j \leq 5)$. Then by Theorem 1 of this paper, one has $r(N) \gg N(\log N)^{-2}$, and so [2, Theorem 1] shows that almost all of the integers thus represented are the sum of 3 cubes and a biquadrate. Consequently N is represented as the sum of 3 cubes and 6 biquadrates.

In order to describe the proof of parts (i) and (ii) of Theorem 5(a) we require some additional notation, and this will be useful also in §8. Take N to be a large integer, and define P_k as in (3.2). We then write $M = P_{21} = P_3^{1/7}$, and define the exponential sum $F(\alpha)$ by

$$F(\alpha) = \sum_{M < p \leq 2M} \sum_{P_3/(2p) \leq x \leq P_3/p} e(\alpha (px)^3).$$

Our Hardy-Littlewood dissection is defined as follows. We put $X = N^{1/100}$, and when $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a, q) = 1$, we write

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) \ : \ |\alpha - a/q| \leq X N^{-1}\}.$$

We take $\mathfrak{M}$ to be the union of the intervals $\mathfrak{M}(q, a)$ with $0 \leq a \leq q \leq X$ and $(a, q) = 1$, and write $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. Note that the intervals $\mathfrak{M}(q, a)$ comprising $\mathfrak{M}$ are pairwise disjoint.

In order to establish the remaining parts of Theorem 5 we require a reasonably strong mean value estimate for the cubic exponential sum $F(\alpha)$.

Lemma 7.1. *We have*

$$\int_{\mathfrak{m}} |F(\alpha)|^8 d\alpha \ll P_3^{5+\varepsilon} X^{-1/3}. \tag{7.1}$$

Proof. We establish the upper bound (7.1) by means of the Hardy-Littlewood method, and begin by recalling some estimates of use on the major and minor arcs. First, by considering the underlying diophantine equations, it follows from [4, Lemma 6] that

$$\int_0^1 |F(\alpha)|^6 d\alpha \ll P_3^{7/2+\varepsilon} M^{-3/2}. \tag{7.2}$$

Moreover, when $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$, $q \leq N^{1/2}$ and $|q\alpha - a| \leq N^{-1/2}$, it follows from [3, Lemma 6] that

$$F(\alpha) \ll P_3^{3/4+\varepsilon} M^{1/4} + P_3^{1+\varepsilon} (q + N|q\alpha - a|)^{-1/3}. \tag{7.3}$$

We next describe the Hardy-Littlewood dissection. Recall the notation concluding §3, and put $Q = (P_3M^{-1})^{3/4}$. Suppose first that $\alpha \in \mathfrak{m}(Q)$. By Dirichlet's Theorem on diophantine approximation, there exist $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q)=1$, $q \le N^{1/2}$ and $|q\alpha - a| \le N^{-1/2}$. But $\alpha \notin \mathfrak{M}(Q)$, so that necessarily one has either $|q\alpha - a| > QN^{-1}$ or else $q > Q$. Then it follows from (7.3) that

$$\sup_{\alpha\in\mathfrak{m}(Q)} |F(\alpha)| \ll P_3^{3/4+\varepsilon} M^{1/4},$$

whence by (7.2),

$$\int_{\mathfrak{m}(Q)} |F(\alpha)|^8 d\alpha \ll \Big(\sup_{\alpha\in\mathfrak{m}(Q)} |F(\alpha)|\Big)^2 \int_0^1 |F(\alpha)|^6 d\alpha \ll P_3^{5+\varepsilon} M^{-1}. \tag{7.4}$$

Next we note that if $\alpha \in \mathfrak{M}(Q)\cap\mathfrak{m}$, then there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$, $q \le Q$ and $|q\alpha - a| \le QN^{-1}$. But since $\alpha \in \mathfrak{m}$, one must have either $|\alpha - a/q| > XN^{-1}$ or else $q > X$. Thus we deduce from (7.3) that

$$\sup_{\alpha\in\mathfrak{M}(Q)\cap\mathfrak{m}} |F(\alpha)| \ll P_3^{1+\varepsilon} X^{-1/3}.$$

Consequently, on noting that when $\alpha \in \mathfrak{M}(Q)$ the second term on the right hand side of (7.3) dominates the first, we obtain

$$\begin{aligned}\int_{\mathfrak{M}(Q)\cap\mathfrak{m}} |F(\alpha)|^8 d\alpha &\ll \Big(\sup_{\mathfrak{M}(Q)\cap\mathfrak{m}} |F(\alpha)|\Big)\int_{\mathfrak{M}(Q)} |F(\alpha)|^7 d\alpha\\ &\ll P_3^{8+\varepsilon}X^{-1/3}\sum_{1\le q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-7/3}\int_0^\infty (1+N\beta)^{-7/3}d\beta\\ &\ll P_3^{5+\varepsilon}X^{-1/3}.\end{aligned} \tag{7.5}$$

The proof of the lemma is completed by combining (7.4) and (7.5).

The proof of part (i) of Theorem 5(*a*). Let N be a large natural number, and write $\mathcal{R}_1(N)$ for the number of representations of N in the form

$$N = \sum_{i=1}^{2}\big((x_i+y_i)^4 + (x_i-y_i)^4 + (2y_i)^4\big) + \sum_{j=1}^{3} z_j^4 + (px)^3, \tag{7.6}$$

with

$$\begin{gathered}P/4 \le x_i, y_i \le P \quad\text{and}\quad x_i \ne y_i \quad (i=1,2), \quad 1 \le z_j \le P \quad (1\le j\le 3),\\ M < p \le 2M \quad\text{and}\quad P_3/(2p) \le x \le P_3/p.\end{gathered} \tag{7.7}$$

We aim to establish that $\mathcal{R}_1(N) > 0$, whence by (7.6) the integer N is represented as the sum of 9 biquadrates and a cube. When $\mathfrak{B} \subseteq [0,1)$, define

$$\mathcal{R}_1(N;\mathfrak{B}) = \int_{\mathfrak{B}} g(\alpha)^2 f_4(\alpha)^3 F(\alpha) e(-N\alpha) d\alpha. \tag{7.8}$$

Then by (7.6), (3.1) and (3.3) we have

$$\mathcal{R}_1(N) = \mathcal{R}_1(N;[0,1)) = \mathcal{R}_1(N;\mathfrak{M}) + \mathcal{R}_1(N;\mathfrak{m}). \tag{7.9}$$

We begin by treating the minor arcs. Applying Hölder's inequality to (7.8) in combination with Lemmata 3.4, 3.5 and 7.1, we obtain

$$\begin{aligned}\mathcal{R}_1(N;\mathfrak{m}) &\le \Big(\int_0^1 |g(\alpha)^2 f_4(\alpha)^4| d\alpha\Big)^{3/4}\Big(\int_0^1 |g(\alpha)|^4 d\alpha\Big)^{1/8}\Big(\int_{\mathfrak{m}} |F(\alpha)|^8 d\alpha\Big)^{1/8}\\ &\ll N^{13/12+\varepsilon} X^{-1/24}.\end{aligned} \tag{7.10}$$

We now turn our attention to the estimation of $\mathcal{R}_1(N;\mathfrak{M})$. Suppose that $\alpha \in \mathbb{R}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$ and $q \le X$, and write $\beta = \alpha - a/q$. We note that when $M < p \le 2M$, one has $p > X$, and so $(ap^3,q)=1$. It therefore follows from [21, Theorem 4.1] that in such circumstances, one has

$$\sum_{P_3/(2p)\le x\le P_3/p} e((px)^3\alpha) - q^{-1}S_3(q,ap^3)\int_{P_3/(2p)}^{P_3/p} e(\beta p^3t^3)\,dt \ll q^{1/2+\varepsilon}(1+N|\beta|)^{1/2}. \tag{7.11}$$

Write

$$\tilde{v}_3(\beta) = \int_{P_3/2}^{P_3} e(\beta t^3)\,dt. \tag{7.12}$$

Then by a change of variable, the integral in (7.11) is equal to $p^{-1}\tilde{v}_3(\beta)$. Moreover, in view of the coprimality of p and q, one has $S_3(q,ap^3)=S_3(q,a)$. Consequently, when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, it follows from (7.11) that

$$F(\alpha) = \Xi q^{-1}S_3(q,a)\tilde{v}_3(\alpha - a/q) + O(MX^{1+\varepsilon}), \tag{7.13}$$

where

$$\Xi = \Xi(M) = \sum_{M<p\le 2M} p^{-1} \gg (\log N)^{-1}. \tag{7.14}$$

For future use, we define the function $T(\alpha)$ by

$$T(\alpha) = \begin{cases} \Xi q^{-1}S_3(q,a)\tilde{v}_3(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}(q,a)\subseteq \mathfrak{M},\\ 0, & \text{otherwise,}\end{cases} \tag{7.15}$$

and modify the definitions (4.6) and (4.7) by defining

$$V(\alpha) = \begin{cases} q^{-1}S_4(q,a)v_4(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}(q,a)\subseteq \mathfrak{M},\\ 0, & \text{otherwise,}\end{cases} \tag{7.16}$$

and

$$W(\alpha) = \begin{cases} q^{-2}S(q,a)v(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}(q,a)\subseteq \mathfrak{M},\\ 0, & \text{otherwise.}\end{cases} \tag{7.17}$$

Recalling the estimates provided by (7.13) and Lemmata 3.1 and 3.3, one has for each $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$ the upper bound

$$g(\alpha)^2f_4(\alpha)^3F(\alpha) - W(\alpha)^2V(\alpha)^3T(\alpha) \ll X^2P^6P_3 + X^{1+\varepsilon}MP^7.$$

Since $\mathfrak{M}$ has measure $O(X^3N^{-1})$, we conclude from (7.8), (7.9) and (7.10) that

$$\mathcal{R}_1(N) - \Xi J_1^*(N)\sum_{1\le q\le X}\mathcal{S}_1(q,N) \ll N^{13/12+\varepsilon}X^{-1/24}, \tag{7.18}$$

where

$$\mathcal{S}_1(q,N) = q^{-8}\sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)^2S_4(q,a)^3S_3(q,a)e(-Na/q), \tag{7.19}$$

and

$$J_1^*(N) = \int_{-X/N}^{X/N} v(\beta)^2v_4(\beta)^3\tilde{v}_3(\beta)e(-N\beta)\,d\beta. \tag{7.20}$$

But on writing

$$J_1(N) = \int_{-\infty}^{\infty} v(\beta)^2v_4(\beta)^3\tilde{v}_3(\beta)e(-N\beta)\,d\beta, \tag{7.21}$$

we find from (7.12) and Lemmata 3.1 and 3.3 that $J_1(N)$ is absolutely convergent, and, moreover, it follows from (7.20) and (7.21) that

$$J_1(N) - J_1^*(N) \ll N^{25/12} \int_{X/N}^{\infty} (1+N\beta)^{-37/12} d\beta \ll N^{13/12} X^{-2}. \tag{7.22}$$

Furthermore, a straightforward application of Fourier's integral formula demonstrates that $J_1(N) \gg N^{13/12}$, so that together with (7.22) we have

$$N^{13/12} \ll J_1(N) \ll N^{13/12}. \tag{7.23}$$

Next write

$$\mathfrak{S}_1(N) = \sum_{q=1}^{\infty} q^{-8} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)^2 S_4(q,a)^3 S_3(q,a) e(-Na/q),$$

and note that by (7.19) together with Lemmata 3.1 and 3.3, we have

$$|\mathcal{S}_1(q,N)| \ll q^{\varepsilon-13/12}, \tag{7.24}$$

so that

$$\mathfrak{S}_1(N) - \sum_{1\le q\le X} \mathcal{S}_1(q,N) \ll \sum_{q>X} |\mathcal{S}_1(q,N)| \ll X^{\varepsilon-1/12}.$$

We may therefore conclude from (7.18), (7.22) and (7.23) that

$$\mathcal{R}_1(N) - \Xi J_1(N)\mathfrak{S}_1(N) \ll N^{13/12+\varepsilon} X^{-1/24}. \tag{7.25}$$

Provided now that we can show that $\mathfrak{S}_1(N) \gg 1$, it will follow from (7.14), (7.23) and (7.25) that $\mathcal{R}_1(N) \gg N^{13/12}(\log N)^{-1}$, and so the proof of part (i) of Theorem 5(a) will be complete. But on noting that for each prime number p the estimate (7.24) yields

$$\sum_{h=0}^{\infty} \mathcal{S}_1(p^h,N) = 1 + O(p^{\varepsilon-13/12}), \tag{7.26}$$

we may apply the argument employed in §4 to analyse the singular series, mutatis mutandis, and thereby obtain $\mathfrak{S}_1(N) \gg 1$. The only detail not already transparent concerns the solubility, for small primes p, of the congruence

$$z_1^4 + z_2^4 + z_3^4 + z_4^3 + 2(z_5^2+3z_6^2)^2 + 2(z_7^2+3z_8^2)^2 \equiv N \pmod{p^\gamma}, \tag{7.27}$$

where $\gamma = \gamma(p)$ again denotes the integer defined in (4.26). But the Cauchy-Davenport Theorem (see [21, Lemma 2.14]) shows that when $p>3$, the number of residue classes modulo p represented by the polynomial

$$y_1^4 + y_2^4 + y_3^4 + w^3,$$

subject to $(y_1,p)=1$, is at least

$$\min\left\{p,\ 3\frac{p-1}{(4,p-1)} + \frac{p-1}{(3,p-1)}\right\} = p.$$

Thus for each N the congruence (7.27) is soluble with $(z_1,p)=1$. The latter conclusion is immediate when $p=3$, and also follows easily when $p=2$ on noting that w^3 represents all of the odd congruence classes modulo 16. Consequently, as in the conclusion of §4, we find that $\sum_{h=0}^{\infty} \mathcal{S}_1(p^h,N)$ is real and positive for every integer N and prime p, and in combination with (7.26) we obtain the desired conclusion that $\mathfrak{S}_1(N) \gg 1$.

The proof of part (ii) of Theorem 5(a) is similar to that of part (i), though in many respects simpler, and we therefore omit the uninteresting details.

The proof of part (ii) of Theorem 5(*a*). Let N be a large natural number, and write $\mathcal{R}_2(N)$ for the number of representations of N in the form

$$N=\sum_{i=1}^{2}\big((x_i+y_i)^4+(x_i-y_i)^4+(2y_i)^4\big)+z_1^4+z_2^4+(p_1w_1)^3+(p_2w_2)^3, \tag{7.28}$$

with x_i,y_i,z_i satisfying (7.7) for $i=1,2$, and

$$M<p_j\le 2M \quad \text{and} \quad P_3/(2p_j)\le w_j\le P_3/p_j \quad (j=1,2).$$

Then by (3.1) and (3.3) we have

$$\mathcal{R}_2(N)=\int_0^1 g(\alpha)^2 f_4(\alpha)^2 F(\alpha)^2 e(-N\alpha)d\alpha. \tag{7.29}$$

We aim to show that $\mathcal{R}_2(N)>0$, whence by (7.28) the integer N is represented as the sum of 8 biquadrates and two cubes. When $\mathfrak{B}\subseteq[0,1)$, define

$$\mathcal{R}_2(N;\mathfrak{B})=\int_{\mathfrak{B}} g(\alpha)^2 f_4(\alpha)^2 F(\alpha)^2 e(-N\alpha)d\alpha. \tag{7.30}$$

Then by (7.28), (7.29), (3.1) and (3.3) we have

$$\mathcal{R}_2(N)=\mathcal{R}_2(N;[0,1))=\mathcal{R}_2(N;\mathfrak{M})+\mathcal{R}_2(N;\mathfrak{m}). \tag{7.31}$$

Observe first that by applying Hölder's inequality to (7.29) in combination with Lemmata 3.4, 3.5 and 7.1, we have

$$\begin{aligned}\mathcal{R}_2(N;\mathfrak{m})&\le\int_{\mathfrak{m}}|g(\alpha)f_4(\alpha)F(\alpha)|^2d\alpha\\&\le\Big(\int_0^1|g(\alpha)^2f_4(\alpha)^4|d\alpha\Big)^{1/2}\Big(\int_0^1|g(\alpha)^4|d\alpha\Big)^{1/4}\Big(\int_{\mathfrak{m}}|F(\alpha)|^8d\alpha\Big)^{1/4}\\&\ll N^{7/6+\varepsilon}X^{-1/12}.\end{aligned} \tag{7.32}$$

Next, on recalling the notation defined by (7.15)-(7.17) and making use of (7.13) and Lemmata 3.1 and 3.3, we deduce that when $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$, one has the upper bound

$$g(\alpha)^2f_4(\alpha)^2F(\alpha)^2-W(\alpha)^2V(\alpha)^2T(\alpha)^2\ll X^2P^5P_3^2+X^{1+\varepsilon}MP_3P^6.$$

Consequently, since $\mathfrak{M}$ has measure $O(X^3N^{-1})$, it follows from (7.30)-(7.32) that

$$\mathcal{R}_2(N)-\Xi^2J_2^*(N)\sum_{1\le q\le X}\mathcal{S}_2(q,N)\ll N^{7/6+\varepsilon}X^{-1/12}, \tag{7.33}$$

where Ξ is defined as in (7.14),

$$\mathcal{S}_2(q,N)=q^{-8}\sum_{\substack{a=1\\(a,q)=1}}^{q}S(q,a)^2S_4(q,a)^2S_3(q,a)^2e(-Na/q),$$

and

$$J_2^*(N)=\int_{-X/N}^{X/N}v(\beta)^2v_4(\beta)^2\tilde{v}_3(\beta)^2e(-N\beta)d\beta.$$

In the present problem we have a cubic summand in place of the biquadratic summand occurring in the argument of the proof of part (i), and this causes more rapid convergence in both $J_i^*(N)$ and

$\mathcal{S}_i(q,N)$ when $i=2$ as compared to the situation with $i=1$. Thus the argument of the proof of part (i) of Theorem 5(a) is readily adapted to establish that

$$N^{7/6} \ll J_2^*(N) \ll N^{7/6}, \tag{7.34}$$

and

$$\mathfrak{S}_2(N) - \sum_{1\le q\le X} \mathcal{S}_2(q,N) \ll X^{\varepsilon-1/6}, \tag{7.35}$$

where we write

$$\mathfrak{S}_2(N) = \sum_{q=1}^{\infty} q^{-8} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)^2 S_4(q,a)^2 S_3(q,a)^2 e(-Na/q).$$

The aforementioned argument is also readily adapted to show that $\mathfrak{S}_2(N) \gg 1$. The only detail which requires verification concerns the solubility, for small primes p, of the congruence

$$z_1^4+z_2^4+z_3^3+z_4^3+2(z_5^2+3z_6^2)^2+2(z_7^2+3z_8^2)^2 \equiv N \pmod{p^\gamma}. \tag{7.36}$$

But when $p>3$, the Cauchy-Davenport Theorem (see [21, Lemma 2.14]) demonstrates that (7.36) is soluble with $(z_1,p)=1$, and moreover such a conclusion may be verified directly when $p=2$ or 3. Thus, as in the argument of the proof of part (i) of Theorem 5(a), we find that $\mathfrak{S}_2(N) \gg 1$, whence by (7.33), (7.34) and (7.35) we may conclude that

$$\mathcal{R}_2(N) \gg N^{7/6}(\log N)^{-2}.$$

This completes the proof of part (ii) of Theorem 5(a).

8. Sums of 4 biquadrates and a cube

Experts will recognise that Theorem 5(b) may be expected to follow directly from the argument of the proof of part (ii) of Theorem 5(a) via a suitable application of Bessel's inequality. In this instance, however, our use of the identity (3.1) leaves us in a ternary additive situation, and consequently the analysis of the associated singular series presents considerable technical complications. We arm ourselves in advance of such skirmishes with some useful technical lemmata.

When q and n are natural numbers, write

$$A(q,n) = q^{-4} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)S_4(q,a)S_3(q,a)e(-na/q). \tag{8.1}$$

Lemma 8.1. *When p is a prime number and h is a natural number, one has*

$$A(p^h,n) \ll hp^{-h/12}.$$

Moreover, when $1\le h\le 12$, one has the potentially sharper estimate

$$A(p^h,n) \ll p^{-1}(p,n)^{1/2}.$$

Proof. The first estimate of the lemma is easily established by applying Lemmata 3.1 and 3.3 to (8.1), thus obtaining

$$A(p^h,n) \ll p^{-4h}\phi(p^h)(hp^{3h/2})(p^{3h/4})(p^{2h/3}) \ll hp^{-h/12}.$$

We next establish the second estimate, noting first that the argument of §6 leading to (6.12) on this occasion yields, for each natural number q, the estimate

$$\phi(q)A(q,n) = \sum_{\substack{l=1\\(l,q)=1}}^{q} A(q,l^{12}n) = q^{-4} \sum_{\substack{a=1\\(a,q)=1}}^{q} S(q,a)S_4(q,a)S_3(q,a)U(q,-na), \tag{8.2}$$

where

$$U(q,b)=\sum_{\substack{l=1\\(l,q)=1}}^{q} e(bl^{12}/q).$$

On applying [11, Lemma 1.2], one finds that the estimate (6.14) remains valid, and thus whenever $(a,p)=1$ one has

$$U(p^h,-na)\ll p^{h/2}(p^h,n)^{1/2}\le p^{h-1/2}(p,n)^{1/2}. \tag{8.3}$$

Moreover, Lemma 3.3 shows that whenever $(a,p)=1$ one has $S(p^h,a)\ll hp^{3h/2}$, and Lemma 3.2 provides the estimates

$$S_4(p^h,a)\ll p^{r_h}\quad\text{and}\quad S_3(p^h,a)\ll p^{s_h}, \tag{8.4}$$

where $r_1=s_1=1/2$, $r_2=s_2=1$, $r_3=s_3=2$, $r_4=3$ and $s_4=5/2$, $r_5=7/2$ and $s_5=3$, and where $r_h=3h/4$ and $s_h=2h/3$ for $h\ge 6$. On substituting (8.3) and (8.4) into (8.2), we obtain

$$A(p^h,n)\ll hp^{t_h}(p,n)^{1/2},$$

where $t_h=r_h+s_h-(3h+1)/2$. When $1\le h\le 5$ it is easily verified by hand that $t_h\le -1$, and when $h\ge 6$ one has

$$t_h=-\frac{h}{12}-\frac{1}{2}\le -1.$$

Thus $t_h\le -1$ for every natural number h, and the second part of the lemma follows immediately.

Armed with the estimate for $A(q,n)$ provided by Lemma 8.1, we next investigate the convergence of a truncated form of the singular series which arises in our subsequent investigations. When n is a natural number and Z is a positive real number, we write

$$T(p,n)=\sum_{h=0}^{\infty}A(p^h,n), \tag{8.5}$$

and

$$\mathcal{P}(n,Z)=\prod_{p\le Z}T(p,n). \tag{8.6}$$

Lemma 8.2. *Let N be a large real number. Then whenever Z is a large real number with $Z\le N$, and n is a natural number with $1\le n\le N$, one has*

$$\mathcal{P}(n,Z)\gg \exp(-\sqrt{\log N}).$$

Proof. We begin by investigating the contribution of the large primes to the product (8.6). By Lemma 8.1, we have

$$\sum_{h=1}^{\infty}|A(p^h,n)|\ll \sum_{h=1}^{11}p^{-1}(p,n)^{1/2}+\sum_{h=12}^{\infty}hp^{-h/12}\ll p^{-1}(p,n)^{1/2},$$

whence by (8.5) there exists an absolute constant B such that

$$|T(p,n)-1|\le Bp^{-1}(p,n)^{1/2}.$$

But the latter estimate yields

$$\prod_{2B^2\le p\le Z}T(p,n)\ge \prod_{2B^2\le p\le Z}(1-Bp^{-1})\prod_{\substack{p\ge 2B^2\\ p|n}}(1-Bp^{-1/2}). \tag{8.7}$$

It follows from Merten's formula (see, for example, [7]) that the first product on the right hand side of (8.7) is $\gg(\log Z)^{-B}$. Meanwhile, on noting that by the Prime Number Theorem one has

$$\sum_{p|n}p^{-1/2}\le \sum_{p\le 2\log N}p^{-1/2}\ll \frac{\sqrt{\log N}}{\log\log N}, \tag{8.8}$$

and moreover that $\log(1-x) > -4x$ for $0 < x \le 1/\sqrt{2}$, we deduce that

$$\prod_{\substack{p\ge 2B^2\\ p|n}} (1 - Bp^{-1/2}) > \exp\Big(-4B\sum_{p|n} p^{-1/2}\Big) \gg \exp(-\varepsilon\sqrt{\log N}).$$

Consequently, we may conclude from (8.7) that

$$\prod_{2B^2\le p\le Z} T(p,n) \gg \exp(-\varepsilon\sqrt{\log N}). \tag{8.9}$$

In order to establish the lower bound for $\mathcal{P}(n,Z)$ claimed in the statement of the lemma, it suffices now to show that $T(p,n) \gg 1$ for the primes p with $p < 2B^2$. But, as in the argument of §6, one has for each natural number H,

$$\sum_{h=0}^{H} A(p^h,n) = p^{-3H} M_n^*(p^H), \tag{8.10}$$

where $M_n^*(p^H)$ denotes the number of solutions of the congruence

$$2(x^2+3y^2)^2 + z^4 + w^3 \equiv n \pmod{p^H}, \tag{8.11}$$

with $1 \le x,y,z,w \le p^H$. Thus we may follow the argument of §6 to show that $M_n^*(p^H) \gg p^{3H}$ provided only that when $H=\gamma$, the congruence (8.11) is soluble with $(z,p)=1$. Moreover the same conclusion $M_n^*(p^H) \gg p^{3H}$ holds also when $p=2$ provided that (8.11) is soluble when $H=\gamma$ with w odd.

Suppose first that $p>3$, and note that the discussion of §6 leading to (6.31) shows that the polynomial x^2+3y^2 represents all residue classes modulo p. Then the Cauchy-Davenport Theorem (see [21, Lemma 2.14]) shows that the number of distinct residue classes represented by the polynomial

$$2(x^2+3y^2)^2 + z^4 + w^3,$$

subject to $(z,p)=1$, is at least $\min\{p,\kappa(p)\}$, where

$$\begin{aligned}\kappa(p) &= \frac{p-1}{2} + \frac{p-1}{(4,p-1)} + \frac{p-1}{(3,p-1)}\\ &= \left(\frac{1}{2} + \frac{1}{(3,p-1)} + \frac{1}{(4,p-1)}\right)(p-1).\end{aligned} \tag{8.12}$$

When $p \ge 13$, therefore, one has

$$\kappa(p) \ge \frac{13}{12}(p-1) \ge p,$$

and moreover one may verify from (8.12) that $\kappa(p) \ge p$ also when $p = 5$, 7 and 11. We may thus conclude that for each natural number n, the congruence (8.11) is soluble when $H=\gamma$ with $(z,p)=1$. Furthermore, when $p=3$ the latter conclusion is essentially trivial, and thus we deduce that $M_n^*(p^H) \gg p^{3H}$ whenever $p>2$. When $p=2$ we note merely that w^3 represents all of the odd congruence classes modulo 16, and so the congruence (8.11) is necessarily soluble with w odd. Thus, as in the argument following (6.32), we again deduce that $M_n^*(2^H) \gg 2^{3H}$. Collecting together the conclusions of this paragraph, we may conclude from (8.10) that

$$T(p,n) = \lim_{H\to\infty} \sum_{h=0}^{H} A(p^h,n) \gg 1$$

for every prime p, whence the conclusion of the lemma follows immediately from (8.9).

Having established a lower bound for a truncated product associated with the singular series, we next investigate a related truncated sum. When n is a natural number and X is a large real number, write

$$\mathfrak{S}(n,X) = \sum_{1\le q\le X} A(q,n). \tag{8.13}$$

Lemma 8.3. *Suppose that N is a large real number, and let $X = N^{1/100}$. Then for all but $O(N\exp(-\sqrt{\log N}))$ of the integers n with $N/2 \le n \le N$, one has*

$$\mathfrak{S}(n,X) \gg \exp(-\sqrt{\log N}).$$

Proof. It follows from the standard theory of exponential sums (see, for example, the proofs of Lemmata 2.10 and 2.11 of [21]) that $A(q,n)$ is a multiplicative function of q. Write $Y = \exp(\sqrt{\log N})$, put $Z = Y^{50}$, and define

$$\mathcal{D} = \mathcal{D}(Z) = \{q \in \mathbb{N} \,:\, p|q \Rightarrow p \le Z\}.$$

Suppose that n is a natural number with $N/2 \le n \le N$. Then on recalling (8.6) and (8.13), we have

$$\mathfrak{S}(n,X) - \mathcal{P}(n,Z) = \mathfrak{S}_1(n) - \mathfrak{S}_2(n), \tag{8.14}$$

where

$$\mathfrak{S}_1(n) = \sum_{\substack{Z<q\le X\\ q\notin\mathcal{D}}} A(q,n) \quad\text{and}\quad \mathfrak{S}_2(n) = \sum_{\substack{q>X\\ q\in\mathcal{D}}} A(q,n). \tag{8.15}$$

We first estimate $\mathfrak{S}_2(n)$. Put $\eta = 300/\sqrt{\log N}$. Then for $q > X$ we have $1 < (q/X)^\eta = q^\eta Y^{-3}$, and thus the multiplicative property of $A(q,n)$ ensures that

$$|\mathfrak{S}_2(n)| < Y^{-3} \sum_{q\in\mathcal{D}} q^\eta |A(q,n)| = Y^{-3} \prod_{p\le Z} \Bigl(\sum_{h=0}^{\infty} p^{h\eta} |A(p^h,n)| \Bigr). \tag{8.16}$$

But when $p \le Z$, Lemma 8.1 yields the estimate

$$\sum_{h=0}^{\infty} p^{h\eta}|A(p^h,n)| - 1 \ll p^{-1+12\eta}(p,n)^{1/2} \ll p^{-1}(p,n)^{1/2},$$

and thus it follows from (8.8) and (8.16) that for some absolute constant B,

$$|\mathfrak{S}_2(n)| < Y^{-3} \prod_{p\le Z}(1+Bp^{-1}) \prod_{p|n}(1+Bp^{-1/2}) \ll Y^{-2}. \tag{8.17}$$

Next we turn our attention to $\mathfrak{S}_1(n)$. For the sake of concision, write

$$\tilde{S}(q,a) = S(q,a)S_4(q,a)S_3(q,a).$$

Also, denote by $\|\beta\|$ the distance between β and the nearest integer. Then by (8.1) and (8.15) we have

$$\sum_{N/2\le n\le N} |\mathfrak{S}_1(n)|^2 = \sum_{\substack{Z<q\le X\\ q\notin\mathcal{D}}} \sum_{\substack{Z<r\le X\\ r\notin\mathcal{D}}} (qr)^{-4} V(q,r), \tag{8.18}$$

where

$$V(q,r) = \sum_{\substack{a=1\\ (a,q)=1}}^{q} \sum_{\substack{b=1\\ (b,r)=1}}^{r} \tilde{S}(q,a)\tilde{S}(r,-b) \sum_{N/2\le n\le N} e\Bigl(\Bigl(\frac{b}{r}-\frac{a}{q}\Bigr)n\Bigr). \tag{8.19}$$

When $q \le X$, $r \le X$ and $a/q \ne b/r$, one has

$$\Bigl\| \frac{a}{q} - \frac{b}{r} \Bigr\| \ge (qr)^{-1} \ge X^{-2},$$

and in such circumstances the innermost sum in (8.19) is $O(X^2)$. Thus we deduce from (8.18) that

$$\begin{aligned}\sum_{N/2\le n\le N} |\mathfrak{S}_1(n)|^2 \ll & N \sum_{Z<q\le X} q^{-8} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |\tilde{S}(q,a)|^2 \\ & + X^2 \Bigl(\sum_{Z<q\le X} q^{-4} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |\tilde{S}(q,a)| \Bigr)^2.\end{aligned}$$

But Lemmata 3.1 and 3.3 provide the estimate $\tilde{S}(q,a) \ll q^{35/12+\varepsilon}$, and so

$$\sum_{N/2\le n\le N} |\mathfrak{S}_1(n)|^2 \ll N\sum_{q>Z} q^{\varepsilon-7/6} + X^2\Big(\sum_{1\le q\le X} q^{\varepsilon-1/12}\Big)^2$$
$$\ll NZ^{\varepsilon-1/6} + X^4 \ll NY^{-5}. \tag{8.20}$$

Note that (8.20) implies that $|\mathfrak{S}_1(n)| \le Y^{-2}$ for all but $O(NY^{-1})$ values of n with $N/2 \le n \le N$. Thus, on collecting together (8.14), (8.17), (8.20), and recalling Lemma 8.2, we deduce that

$$\mathfrak{S}(n,X) \gg \exp(-\sqrt{\log N}) + O(Y^{-2})$$

for all but $O(NY^{-1})$ values of n with $N/2 \le n \le N$. This completes the proof of the lemma.

Our analysis of the truncated singular series now complete, we may swiftly dispose of the proof of Theorem 5(b).

The proof of Theorem 5(*b*). Let N be a large real number, let n be a natural number with $N/2 \le n \le N$, and let $\mathcal{R}_3(n)$ denote the number of representations of n in the form

$$n = (x+y)^4 + (x-y)^4 + (2y)^4 + z^4 + (pw)^3, \tag{8.21}$$

with

$$P/4 \le x, y \le P \quad \text{and} \quad x \ne y, \quad 1 \le z \le P,$$
$$M < p \le 2M \quad \text{and} \quad P_3/(2p) \le w \le P_3/p.$$

We will show that $\mathcal{R}_3(n) > 0$ for each n with $N/2 \le n \le N$, save for at most $O(N\exp(-\sqrt{\log N}))$ possible exceptions. By summing over dyadic intervals, it follows from the latter assertion, together with (8.21), that almost all positive integers are the sum of four biquadrates and a cube, whence Theorem 5(b) follows. When $\mathfrak{B} \subseteq [0,1)$, define

$$\mathcal{R}_3(n;\mathfrak{B}) = \int_{\mathfrak{B}} g(\alpha)f_4(\alpha)F(\alpha)e(-n\alpha)d\alpha. \tag{8.22}$$

Then by (8.21), (3.1) and (3.3) we have

$$\mathcal{R}_3(n) = \mathcal{R}_3(n;[0,1)) = \mathcal{R}_3(n;\mathfrak{M}) + \mathcal{R}_3(n;\mathfrak{m}). \tag{8.23}$$

We first treat the minor arcs $\mathfrak{m}$, noting that an application of Bessel's inequality combined with (7.32) yields

$$\sum_{N/2\le n\le N} |\mathcal{R}_3(n;\mathfrak{m})|^2 \le \int_{\mathfrak{m}} |g(\alpha)f_4(\alpha)F(\alpha)|^2 d\alpha \ll N^{7/6+\varepsilon}X^{-1/12}. \tag{8.24}$$

Next, on recalling the notation defined by (7.15)-(7.17), and making use of (7.13) and Lemmata 3.1 and 3.3, we deduce that when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, one has the upper bound

$$g(\alpha)f_4(\alpha)F(\alpha) - W(\alpha)V(\alpha)T(\alpha) \ll X^2P^2P_3 + X^{1+\varepsilon}MP^3.$$

Then since $\mathfrak{M}$ has measure $O(X^3N^{-1})$, it follows from (8.22)-(8.24) that

$$\sum_{N/2\le n\le N} |\mathcal{R}_3(n) - \Xi J_3^*(n)\mathfrak{S}(n,X)|^2 \ll N^{7/6+\varepsilon}X^{-1/12}, \tag{8.25}$$

where Ξ and $\mathfrak{S}(n,X)$ are defined, respectively, as in (7.14) and (8.13), and

$$J_3^*(n) = \int_{-X/N}^{X/N} v(\beta)v_4(\beta)\tilde{v}_3(\beta)e(-n\beta)d\beta. \tag{8.26}$$

Write

$$J_3(n) = \int_{-\infty}^{\infty} v(\beta)v_4(\beta)\tilde{v}_3(\beta)e(-n\beta)d\beta.$$

Then by (7.12) and Lemmata 3.1 and 3.3, one has

$$J_3(n) - J_3^*(n) \ll N^{13/12} \int_{X/N}^{\infty} (1+N\beta)^{-19/12} d\beta \ll N^{1/12}X^{-1/2}. \tag{8.27}$$

Furthermore, a straightforward application of Fourier's integral formula demonstrates that $J_3(n) \gg N^{1/12}$, so that together with (8.27) we have

$$N^{1/12} \ll J_3(n) \ll N^{1/12}. \tag{8.28}$$

But the inequality (8.25) shows that for every n satisfying $N/2 \le n \le N$, with at most $O(N\exp(-\sqrt{\log N}))$ exceptions, one has

$$\mathcal{R}_3(n) - \Xi J_3^*(n)\mathfrak{S}(n,X) \ll N^{1/12}X^{-1/25}.$$

Thus, since by (8.27) and (8.28) one has

$$J_3^*(n) \gg N^{1/12} + O(N^{1/12}X^{-1/2}),$$

and since by Lemma 8.3, one has for every n satisfying $N/2 \le n \le N$, with at most $O(N\exp(-\sqrt{\log N}))$ exceptions, the lower bound

$$\mathfrak{S}(n,X) \gg \exp(-\sqrt{\log N}),$$

we may conclude that for every integer n satisfying $N/2 \le n \le N$, with at most $O(N\exp(-\sqrt{\log N}))$ exceptions, one has

$$\mathcal{R}_3(n) \gg N^{1/12}\exp(-2\sqrt{\log N}).$$

Consequently the assertion made in the opening paragraph of this proof does indeed hold, and so the proof of Theorem 5(b) is complete.

9. An application to a problem with prime variables

Our objective in this section is the proof of Theorem 7. Since the central variables under consideration will now be prime numbers, it is necessary to introduce some additional notation. We take N to be the large real parameter introduced in §3, and define the generating functions

$$f_k^*(\alpha) = \sum_{1<p\le P_k} e(\alpha p^k) \quad \text{and} \quad g^*(\alpha) = \sum_{\substack{1\le m\le \sqrt{N}/3\\ m\in\mathcal{C}}} e(2m^2\alpha), \tag{9.1}$$

where the first summation is over prime numbers, and $\mathcal{C}$ is the set of integers defined in (2.5). We require an approximation to $f_k^*(\alpha)$ on the major arcs of a Hardy-Littlewood dissection, and this is supplied in all essentials by Hua [12]. When $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta \in \mathbb{R}$, we write

$$S_k^*(q,a) = \sum_{\substack{r=1\\(r,q)=1}}^{q} e(ar^k/q) \quad \text{and} \quad v_k^*(\beta) = \int_2^{P_k} e(\beta t^k)\frac{dt}{\log t}. \tag{9.2}$$

Lemma 9.1. *Suppose that $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta \in \mathbb{R}$, and that $(a,q)=1$. Then*

$$S_k^*(q,a) \ll q^{1/2+\varepsilon} \quad \textit{and} \quad v_k^*(\beta) \ll P_k(1+N|\beta|)^{-1/k}.$$

Suppose further that $\alpha \in \mathbb{R}$, and that for some fixed positive number A, one has $|q\alpha - a| \le (\log N)^A N^{-1}$ and $q \le (\log N)^A$. Then

$$f_k^*(\alpha) = \phi(q)^{-1}S_k^*(q,a)v_k^*(\alpha - a/q) + O(P_k\exp(-c\sqrt{\log N}))$$

for some $c>0$, where here we write $\phi(q)$ for Euler's totient function.

Proof. The lemma follows immediately from Lemmata 7.14-7.16 and 8.5 of [12].

Before fully engaging the proof, it is useful also to record a lower bound for an auxiliary singular series. When m is a natural number, write

$$\mathfrak{S}_k^*(m)=\sum_{q=1}^{\infty}\mathcal{S}_k^*(q,m), \tag{9.3}$$

where

$$\mathcal{S}_k^*(q,m)=\phi(q)^{-5}\sum_{\substack{a=1\\(a,q)=1}}^{q}S_4^*(q,a)^4S_k^*(q,a)e(-ma/q). \tag{9.4}$$

Further, define the sets $\mathcal{M}^{(i)}$ for $i=1,2,3$ by

$$\mathcal{M}^{(1)}=\{m\in\mathbb{N}\ :\ m\equiv 1 \pmod 2,\, m\not\equiv 1 \pmod 3 \text{ and } m\not\equiv -1 \pmod 5\},$$

$$\mathcal{M}^{(2)}=\{m\in\mathbb{N}\ :\ m\equiv 5 \pmod 8,\, m\equiv 2 \pmod 3 \text{ and } m\equiv 0 \text{ or } 3 \pmod 5\},$$

$$\mathcal{M}^{(3)}=\{m\in\mathbb{N}\ :\ m\equiv 5 \pmod{16},\, m\equiv 2 \pmod 3 \text{ and } m\equiv 0 \pmod 5\},$$

and when k is a natural number, define the set $\mathcal{M}_k^*$ by

$$\mathcal{M}_k^*=\begin{cases}\mathcal{M}^{(1)}, & \text{when } k \text{ is odd},\\ \mathcal{M}^{(2)}, & \text{when } 2\|k,\\ \mathcal{M}^{(3)}, & \text{when } 4|k.\end{cases}$$

Lemma 9.2. *For each natural number m one has $\mathfrak{S}_k^*(m)\ge 0$. Moreover, whenever m is a natural number with $m\in\mathcal{M}_k^*$ and $m\not\equiv 1 \pmod{13}$, one has $\mathfrak{S}_k^*(m)\gg 1$.*

Proof. We note first that by the standard theory of exponential sums, one has that $\mathcal{S}_k^*(q,m)$ is a multiplicative function of q (see [12, Lemma 8.1]). Moreover, on recalling the notation introduced in (4.26), it follows from [12, Lemma 8.3] that $S_4^*(p^h,a)=0$ when $(p,a)=1$ and $h>\gamma(p)$. Thus $\mathcal{S}_k^*(p^h,m)=0$ for $h>\gamma(p)$, and so it follows from (9.3) that

$$\mathfrak{S}_k^*(m)=\prod_p T_k^*(p,m), \tag{9.5}$$

where

$$T_k^*(p,m)=\sum_{h=0}^{\gamma(p)}\mathcal{S}_k^*(p^h,m). \tag{9.6}$$

Furthermore, on writing $M_{k,m}^*(p)$ for the number of solutions of the congruence

$$x_1^4+x_2^4+x_3^4+x_4^4+x_5^k\equiv m \pmod{p^\gamma}, \tag{9.7}$$

with $1\le x_j\le p^\gamma$ and $(x_j,p)=1$ $(1\le j\le 5)$, it follows from (9.6) that

$$T_k^*(p,m)=p^\gamma\phi(p^\gamma)^{-5}M_{k,m}^*(p). \tag{9.8}$$

In particular, therefore, one has for each prime p that $T_k^*(p,m)$ is real and non-negative, whence $\mathfrak{S}_k^*(m)\ge 0$. This completes the proof of the first assertion of the lemma.

We next dispose of the contribution of the large primes to $\mathfrak{S}_k^*(m)$. By applying the estimate supplied by Lemma 9.1 within (9.4) and (9.6), one has

$$T_k^*(p,m)=1+O(p^{\varepsilon-3/2}).$$

Thus there exists an absolute constant C such that

$$\tfrac{1}{2} \le \prod_{p \ge C} T_k^*(p,m) \le 2. \tag{9.9}$$

In view of (9.5), therefore, it suffices to consider only the primes p with $p < C$. The primes p with $p \equiv 3 \pmod 4$ satisfying $7 \le p < C$ may be dealt with via the Cauchy-Davenport Theorem (see [21, Lemma 2.14]). Thus it may be shown that the number of distinct residue classes modulo p represented by the polynomial $x_1^4 + x_2^4 + x_3^4 + x_4^4$, with $(x_j, p) = 1$ $(1 \le j \le 4)$, is at least

$$\min\left\{ p\ ,\ 4\frac{p-1}{(4,p-1)} - 3 \right\} = \min\{p, 2p-5\} = p.$$

One therefore has, for each natural number m, the lower bound $M_{k,m}^*(p) \ge 1$, whence

$$T_k^*(p,m) \ge p(p-1)^{-5} \gg 1. \tag{9.10}$$

Next suppose that p is a prime number with $p \equiv 1 \pmod 4$. In this case we apply exponential sums, noting that for each integer x with $(x,p) = 1$, the number of solutions of the congruence $x^4 \equiv y^4 \pmod p$, with $1 \le y \le p-1$, is precisely 4. Thus, by orthogonality,

$$\begin{aligned}\sum_{a=1}^{p-1} |S_4^*(p,a)|^2 &= \sum_{a=1}^{p} |S_4^*(p,a)|^2 - (p-1)^2 \\ &= 4p(p-1) - (p-1)^2 = (3p+1)(p-1). \end{aligned} \tag{9.11}$$

Moreover, when $(a,p) = 1$, a trivial estimate yields

$$|S_k^*(p,a)| \le p - 1, \tag{9.12}$$

and further, Lemma 3.2 provides the upper bound

$$|S_4^*(p,a)| = |S_4(p,a) - 1| \le 3\sqrt{p} + 1. \tag{9.13}$$

We therefore deduce from (9.4), (9.6) and (9.11)-(9.13) that

$$\begin{aligned}|T_k^*(p,m) - 1| &\le (p-1)^{-5} \sum_{a=1}^{p-1} |S_4^*(p,a)^4 S_k^*(p,a)| \\ &\le (p-1)^{-4}(3\sqrt{p}+1)^2 \sum_{a=1}^{p-1} |S_4^*(p,a)|^2 \\ &\le (p-1)^{-3}(3\sqrt{p}+1)^2(3p+1).\end{aligned}$$

A modest computation leads from here to the upper bound $|T_k^*(p,m) - 1| < 9/10$ whenever $p \ge 37$, whence, under the same conditions,

$$T_k^*(p,m) > 1/10. \tag{9.14}$$

It remains to consider the primes $p = 2, 3, 5, 13, 17$ and 29. Note first that when $p \nmid m$, the congruence class m is represented by the polynomial

$$\phi(\mathbf{x}) = x_1^4 + x_2^4 + x_3^4 + x_4^4,$$

with $(x_j, p) = 1$ $(1 \le j \le 4)$, if and only if the congruence

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 \equiv m x_0^4 \pmod p$$

is soluble with $(x_j, p) = 1$ $(0 \le j \le 4)$. Thus we may apply an exponential sum argument, similar to that described in the previous paragraph, to show that each non-zero residue class modulo p is represented by $\phi(\mathbf{x})$, with $(x_j, p) = 1$ $(1 \le j \le 4)$, provided only that

$$(p-1)^5 > (3\sqrt{p}+1)^3(3p+1)(p-1).$$

In particular, therefore, each non-zero residue class modulo 29 is represented by $\phi(\mathbf{x})$ in the desired manner. Moreover, since 1, 7 and -9 are each biquadratic residues modulo 29, one finds that $\phi(\mathbf{x})$ represents the zero residue class modulo 29, with $(x_j, 29) = 1$ $(1 \le j \le 4)$. Consequently, for every natural number m one has $M^*_{k,m}(29) \ge 1$, whence by (9.8) it follows that the lower bound (9.10) holds also when $p = 29$.

Next, on noting that ± 1 and ± 4 are biquadratic residues modulo 17, one may verify directly that every residue class modulo 17 is represented by $\phi(\mathbf{x})$ with $(x_j, 17) = 1$ $(1 \le j \le 4)$. Consequently, for every natural number m one has $M^*_{k,m}(17) \ge 1$, whence (9.10) holds also for $p = 17$. Also, since 1, 3 and 9 are biquadratic residues modulo 13, one may verify directly that every non-zero residue class is represented by $\phi(\mathbf{x})$ with $(x_j, 13) = 1$ $(1 \le j \le 4)$. Thus, when $p = 13$ the congruence (9.7) is soluble with $(x_j, 13) = 1$ $(1 \le j \le 5)$ for every natural number m, except possibly when $m \equiv 1 \pmod{13}$. It follows that whenever $m \not\equiv 1 \pmod{13}$ one has $M^*_{k,m}(13) \ge 1$, whence (9.10) holds for $p = 13$.

Finally we consider the primes 2, 3 and 5. Recall the definition (4.26). Then when $m \in \mathcal{M}^*_k$ one may verify directly for $p = 2$, 3 and 5 that the congruence (9.7) is soluble with $(x_j, p) = 1$ $(1 \le j \le 5)$. Thus one has $M^*_{k,m}(p) \ge 1$ for $p = 2$, 3 and 5 whenever $m \in \mathcal{M}^*_k$, whence (9.10) holds for these primes p.

On combining the conclusions of the previous three paragraphs together with (9.10) and (9.14), we conclude from (9.5) and (9.9) that $\mathfrak{S}^*_k(m) \gg 1$ provided only that $m \not\equiv 1 \pmod{13}$ and $m \in \mathcal{M}^*_k$. This completes the proof of the lemma.

Before proceeding with the main body of our argument, it is convenient to record a lower bound for a counting function related to one employed in the proof of Theorem 6 in §2. When $t \in \mathbb{N}$ and $a, b \in \mathbb{Z}$, denote by $\rho(m; t; a, b)$ the number of representations of m in the form $m = x^2 + xy + y^2$, with x and y prime numbers satisfying the condition that $(x+y)/2$ is prime, and with $x \equiv a \pmod{t}$ and $y \equiv b \pmod{t}$. It is useful also to define the set $\mathcal{C}(t; a, b)$ by

$$\mathcal{C}(t; a, b) = \{m \in \mathbb{N} \,:\, \rho(m; t; a, b) > 0\}.$$

Lemma 9.3. *Suppose that $t \in \mathbb{N}$ and $a, b \in \mathbb{Z}$ satisfy $(ab(a+b), t) = 1$. Then when x is sufficiently large in terms of t, one has*

$$\sum_{\substack{1 \le m \le x \\ \rho(m;t;a,b) > 0}} 1 \gg_t x(\log x)^{-7}.$$

Proof. Recall the notation of the proof of Theorem 6 in §2, and suppose that t, a and b satisfy the hypotheses of the statement of the lemma. Write $\rho^*(m)$ for $\rho(m; t; a, b)$. Then on noting (2.8), one plainly has

$$\sum_{1 \le m \le x} \rho^*(m)^2 \le \sum_{1 \le m \le x} \rho(m)^2 \le R(x^{1/2}) \ll x \log x. \tag{9.15}$$

But, as a modest elaboration of the lower bound (2.7) evidently within the compass of the Hardy-Littlewood method,

$$\sum_{1 \le m \le x} \rho^*(m) \ge \sum_{\substack{h \text{ prime} \\ 1 \le h \le \frac{1}{2}x^{1/2}}} \sum_{\substack{p_1, p_2 \text{ primes} \\ p_1 \equiv a \pmod{t} \\ p_2 \equiv b \pmod{t} \\ p_1 + p_2 = 2h}} 1 \gg_t x(\log x)^{-3}. \tag{9.16}$$

Then on combining Cauchy's inequality with (9.15) and (9.16), one arrives at the conclusion

$$\sum_{\substack{1 \le m \le x \\ \rho^*(m) > 0}} 1 \ge \Big(\sum_{1 \le m \le x} \rho^*(m) \Big)^2 \Big(\sum_{1 \le m \le x} \rho^*(m)^2 \Big)^{-1} \gg x(\log x)^{-7}.$$

This completes the proof of the lemma.

We note that in our application of Lemma 9.3 in the sequel we take $t = 13$, and thus the implicit constants arising in the lower bound recorded in the statement of Lemma 9.3 are of no importance in our subsequent deliberations.

The conditions are now favourable for us to embark on the proof of Theorem 7. Let k be a fixed natural number with $k \geq 2$, let $\mathcal{M}_k$ be the set of integers defined in the statement of Theorem 7, and let n be an integer with $n \in \mathcal{M}_k \cap [N/2, N]$. Write $\mathcal{R}_k^*(n)$ for the number of representations of n in the form

$$n = 2m_1^2 + 2m_2^2 + \sum_{i=1}^{4} p_i^4 + p_5^k,$$

with $m_1, m_2 \in \mathcal{C} \cap [1, \sqrt{N}/3]$, and with p_j a prime number for $1 \leq j \leq 5$. In view of the identity (1.3) and the definition of the set $\mathcal{C}$, it follows that whenever $\mathcal{R}_k^*(n) > 0$, the integer n possesses a representation in the form (1.2). We put $L = (\log N)^\sigma$, with $\sigma = 2^{6k+7}$, and define $\mathfrak{M} = \mathfrak{M}(L)$ and $\mathfrak{m} = \mathfrak{m}(L)$ as in the concluding paragraph of §3. When $\mathfrak{B} \subseteq [0, 1)$, write

$$\mathcal{R}_k^*(n; \mathfrak{B}) = \int_{\mathfrak{B}} g^*(\alpha)^2 f_4^*(\alpha)^4 f_k^*(\alpha) e(-n\alpha) d\alpha. \tag{9.17}$$

Then by (9.1) we have

$$\mathcal{R}_k^*(n) = \mathcal{R}_k^*(n; [0, 1)) = \mathcal{R}_k^*(n; \mathfrak{M}) + \mathcal{R}_k^*(n; \mathfrak{m}). \tag{9.18}$$

We begin by estimating the contribution of the minor arcs $\mathfrak{m}$. According to [12, Theorem 10], one has

$$\sup_{\alpha \in \mathfrak{m}} |f_k^*(\alpha)| \ll P_k (\log N)^{-100}. \tag{9.19}$$

On considering the diophantine equation underlying the mean value estimate (2.4), moreover, one has

$$\int_0^1 |g^*(\alpha)^2 f_4^*(\alpha)^4| d\alpha \ll N (\log N)^\varepsilon \tag{9.20}$$

(an estimate which may be compared to that recorded in Lemma 3.4). Thus, on combining (9.19) and (9.20) to estimate $\mathcal{R}_k^*(n; \mathfrak{m})$, we deduce from (9.17) that

$$\begin{aligned} \mathcal{R}_k^*(n; \mathfrak{m}) &\leq \sup_{\alpha \in \mathfrak{m}} |f_k^*(\alpha)| \int_0^1 |g^*(\alpha)^2 f_4^*(\alpha)^4| d\alpha \\ &\ll N^{1+1/k} (\log N)^{-99}. \end{aligned} \tag{9.21}$$

We estimate the major arc contribution $\mathcal{R}_k^*(n; \mathfrak{M})$ by considering the tame major arc integral

$$\mathcal{T}_k^*(m) = \int_{\mathfrak{M}} f_4^*(\alpha)^4 f_k^*(\alpha) e(-m\alpha) d\alpha, \tag{9.22}$$

noting that by (9.1) one has

$$\mathcal{R}_k^*(n; \mathfrak{M}) = \sum_{\substack{1 \leq m_1 \leq \sqrt{N}/3 \\ m_1 \in \mathcal{C}}} \sum_{\substack{1 \leq m_2 \leq \sqrt{N}/3 \\ m_2 \in \mathcal{C}}} \mathcal{T}_k^*(n - 2m_1^2 - 2m_2^2). \tag{9.23}$$

When l is a natural number, define the function $V_l^*(\alpha)$ by

$$V_l^*(\alpha) = \begin{cases} \phi(q)^{-1} S_l^*(q, a) v_l^*(\alpha - a/q), & \text{when } \alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}, \\ 0, & \text{otherwise.} \end{cases}$$

Then by Lemma 9.1, one has for each $\alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}$ the upper bound

$$f_4^*(\alpha)^4 f_k^*(\alpha) - V_4^*(\alpha)^4 V_k^*(\alpha) \ll N^{1+1/k} \exp(-c\sqrt{\log N}),$$

for a suitable $c > 0$. Since the measure of $\mathfrak{M}$ is $O((\log N)^{2\sigma} N^{-1})$, we deduce from (9.22) that for each natural number m,

$$\mathcal{T}_k^*(m) - \int_0^1 V_4^*(\alpha)^4 V_k^*(\alpha) e(-m\alpha) d\alpha \ll N^{1/k} (\log N)^{-100},$$

whence

$$\mathcal{T}_k^*(m) = \sum_{1 \le q \le L} \mathcal{S}_k^*(q, m) J_k^*(q, m; N, L) + O(N^{1/k} (\log N)^{-100}), \tag{9.24}$$

where $\mathcal{S}_k^*(q, m)$ is defined as in (9.4), and

$$J_k^*(q, m; N, L) = \int_{-q^{-1}LN^{-1}}^{q^{-1}LN^{-1}} v_4^*(\beta)^4 v_k^*(\beta) e(-\beta m) d\beta. \tag{9.25}$$

Consider a fixed integer m with $N/18 \le m \le N$. On writing

$$J_k^*(m) = \int_{-\infty}^{\infty} v_4^*(\beta)^4 v_k^*(\beta) e(-\beta m) d\beta, \tag{9.26}$$

we deduce from (9.25) and Lemma 9.1 that whenever $1 \le q \le L$, one has

$$\begin{aligned} J_k^*(m) - J_k^*(q, m; N, L) &\ll N^{1+1/k} \int_{q^{-1}LN^{-1}}^{\infty} (1 + N\beta)^{-1-1/k} d\beta \\ &\ll N^{1/k} (q/L)^{1/(2k)}. \end{aligned} \tag{9.27}$$

Furthermore, in view of Lemma 9.1 and (9.26), a straightforward application of Fourier's integral formula demonstrates that $J_k^*(m) \gg N^{1/k} (\log N)^{-5}$. Thus it follows from (9.25) and Lemma 9.1 that

$$N^{1/k} (\log N)^{-5} \ll J_k^*(m) \ll N^{1/k}. \tag{9.28}$$

We next handle the truncated singular series. Recall the definition (9.3). Then by (9.4) and Lemma 9.1 one has

$$\mathfrak{S}_k^*(m) - \sum_{1 \le q \le L} \mathcal{S}_k^*(q, m) \ll \sum_{q > L} q^{\varepsilon - 3/2} \ll L^{-1/3}.$$

Moreover, similarly,

$$\sum_{1 \le q \le L} q^{1/(2k)} |\mathcal{S}_k^*(q, m)| \ll \sum_{1 \le q \le L} q^{-1-1/(4k)} \ll 1.$$

Consequently, on recalling (9.24), (9.27) and (9.28), we may conclude that

$$\mathcal{T}_k^*(m) - J_k^*(m) \mathfrak{S}_k^*(m) \ll N^{1/k} (\log N)^{-100}. \tag{9.29}$$

We now recall (9.18), (9.21) and (9.23), and by means of (9.29) deduce that

$$\mathcal{R}_k^*(n) = \mathcal{U}_k(n) + O(N^{1+1/k} (\log N)^{-99}), \tag{9.30}$$

where

$$\mathcal{U}_k(n) = \sum_{\substack{1 \le m_1 \le \sqrt{N}/3 \\ m_1 \in \mathcal{C}}} \sum_{\substack{1 \le m_2 \le \sqrt{N}/3 \\ m_2 \in \mathcal{C}}} \mathfrak{S}_k^*(n - 2m_1^2 - 2m_2^2) J_k^*(n - 2m_1^2 - 2m_2^2). \tag{9.31}$$

On the one hand we have $n \in \mathcal{M}_k \cap [N/2, N]$, so that for each m_1 and m_2 in the latter summations, one has

$$N/18 \le n - 2m_1^2 - 2m_2^2 \le N.$$

Thus we deduce from (9.28) that whenever m_1 and m_2 occur in the summation of (9.31), one has

$$J_k^*(n - 2m_1^2 - 2m_2^2) \gg N^{1/k} (\log N)^{-5}. \tag{9.32}$$

On the other hand, on observing that whenever $m \in \mathcal{C}$, one has $2m^2 \equiv 18 \pmod{240}$, we find that whenever $n \in \mathcal{M}_k$ and $m_1, m_2 \in \mathcal{C}$, then it follows that $n - 2m_1^2 - 2m_2^2 \in \mathcal{M}_k^*$. Then it follows from Lemma 9.2 that whenever $m_1, m_2 \in \mathcal{C}(13; 1, 1)$ and $n \not\equiv 11 \pmod{13}$, then one has

$$\mathfrak{S}_k^*(n - 2m_1^2 - 2m_2^2) \gg 1, \tag{9.33}$$

and, moreover, whenever $m_1, m_2 \in \mathcal{C}(13; 2, 2)$ and $n \equiv 11 \pmod{13}$, then again it follows from Lemma 9.2 that the lower bound (9.33) holds. On substituting (9.32) and (9.33) into (9.31), we conclude from Lemma 9.3 that whenever $n \in \mathcal{M}_k \cap [N/2, N]$ and $n \not\equiv 11 \pmod{13}$, then

$$\mathcal{U}_k(n) \gg \Bigg(\sum_{\substack{1 \le m \le \sqrt{N}/3 \\ m \in \mathcal{C}(13;1,1)}} 1 \Bigg)^2 N^{1/k} (\log N)^{-5} \gg N^{1+1/k} (\log N)^{-19}, \tag{9.34}$$

and similarly, whenever $n \in \mathcal{M}_k \cap [N/2, N]$ and $n \equiv 11 \pmod{13}$, then

$$\mathcal{U}_k(n) \gg \Bigg(\sum_{\substack{1 \le m \le \sqrt{N}/3 \\ m \in \mathcal{C}(13;2,2)}} 1 \Bigg)^2 N^{1/k} (\log N)^{-5} \gg N^{1+1/k} (\log N)^{-19}. \tag{9.35}$$

On collecting together (9.30), (9.34) and (9.35), we finally deduce that whenever $n \in \mathcal{M}_k \cap [N/2, N]$, one has

$$\mathcal{R}_k^*(n) \gg N^{1+1/k} (\log N)^{-19}.$$

The conclusion of Theorem 7 follows immediately.

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